PRINCIPAL BUNDLES UNDER A REDUCTIVE GROUP

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Abstract. Let k be a field of characteristic 0. We consider principal bundles over a k-scheme with reductive structure group (not necessarily of finite type). It is shown in particular that for k algebraically closed there exists on any complete connected k-scheme a universal such bundle. As a consequence, an explicit description of principal bundles with reductive structure group over curves of genus 0 or 1 is obtained.

1. Introduction

In 1957 Grothendieck [Gro57] showed that if G is a complex Lie group with reductive Lie algebra, then principal bundles over the Riemann sphere with structure group G are classified by conjugacy classes of homomorphisms from the complex multiplicative group to G. This result is equivalent to a purely algebraic one in which the Riemann sphere is replaced by the projective line over $\mathbf{C}$ and G by a reductive algebraic group over $\mathbf{C}$, and indeed $\mathbf{C}$ may be replaced by an arbitrary algebraically closed field k of characteristic 0. Modulo the classification of *vector* bundles on the projective line, Grothendieck's result can be expressed as follows: the functor $H^1(\mathbf{P}^1, -)$ on reductive k-groups up to conjugacy is representable. In this paper it will be shown that an analogous representability result holds in much greater generality, and in particular with $\mathbf{P}^1$ replaced by an arbitrary complete connected k-scheme X.

To state the results more precisely, we need some definitions. Let k be a field and G be an affine k-group, i.e. an affine group scheme over k. We define a principal G-bundle over a k-scheme X by requiring local triviality for the $fpqc$-topology. The set of isomorphism classes of such bundles over X will be written $H^1(X, G)$. When G is of finite type and k is of characteristic 0, local triviality for the $fpqc$-topology is equivalent to that for the étale topology. We may regard $H^1(X, G)$ as a functor on the category of affine k-groups up to conjugacy, in which a morphism from G' to G is an equivalence class of homomorphisms of k-groups from G' to G under the action by conjugation of $G(k)$. Recall that G is the filtered limit of its affine quotient k-groups of finite type. We say that G is *reductive* if each such quotient is reductive in the usual sense. Reductive k-groups are not required to be connected. We now have the following result.

Theorem 1.1. *Let X be a scheme over an algebraically closed field k of characteristic 0 for which $H^0(X, \mathcal{O}_X)$ is a henselian local ring with residue field k. Then the functor $H^1(X, -)$ on the category of reductive k-groups up to conjugacy is representable.*

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Explicitly, Theorem 1.1 states that there exists a reductive k-group G_0 such that we have a bijection

$$\mathrm{Hom}(G_0, G)/G(k) \xrightarrow{\sim} H^1(X, G) \tag{1.1}$$

which is natural in the reductive k-group G. Such a G_0 is of course unique up to isomorphism. Taking the general linear group for G in (1.1) gives a bijection from the set of isomorphism classes of representations of G_0 to the set of isomorphism classes of vector bundles on X. Since vector bundles in general have moduli while representations reductive groups of finite type do not, the k-group G_0 thus cannot in general be of finite type. It also follows from (1.1) that the largest profinite quotient of G_0 is the algebraic fundamental group of X. The k-group G_0 may be regarded as an algebraic analogue of the loop space of a topological space, as it plays a role formally analogous to that of the loop space in the homotopy category (see [Las56] and [Mil56]).

It should be noted that some condition on X is necessary for Theorem 1.1 to hold. It follows for example from (1.1) that the Krull-Schmidt theorem must hold for vector bundles over X.

For $X = \mathbf{P}^1$, the classification of vector bundles on X shows that the universal reductive k-group G_0 is $\mathbf{G}_m$, and we recover the algebraic form of Grothendieck's Theorem.

For X an elliptic curve over k, the universal reductive k-group G_0 is the product of SL_2 with a central extension of k-groups of multiplicative type. To describe it explicitly, write $D(M)$ for the k-group of multiplicative type with character group M. Then $D(\mathbf{Q})$ contains the k-subgroup

$$\lim_n \mu_n = D(\mathbf{Q}/\mathbf{Z}).$$

The embedding of the torsion subgroup of $\mathrm{Pic}^0(X)$ defines a k-quotient of $D(\mathrm{Pic}^0(X))$ which may be identified with the algebraic fundamental group

$$\lim_n \operatorname{Ker} n_X$$

of X, where n_X is the multiplication by n on X. We have a bilinear alternating morphism

$$u_X : D(\mathrm{Pic}^0(X)) \times_k D(\mathrm{Pic}^0(X)) \to D(\mathbf{Q})$$

over k, given by

$$u_X(a, a') = e(\overline{a}, \overline{a}')^{-\frac{1}{2}},$$

where the bar denotes the image in $\lim_n \operatorname{Ker} n_X$,

$$e = \lim_n \widetilde{e}_n : \lim_n \operatorname{Ker} n_X \times_k \lim_n \operatorname{Ker} n_X \to \lim_n \mu_n$$

with $\widetilde{e}_n$ the Weil pairing, and the square root in $D(\mathbf{Q})$ is well-defined because $D(\mathbf{Q})$ is uniquely 2-divisible. Now given commutative affine k-groups G' and G' and a bilinear alternating morphism

$$z : G' \times_k G' \to G'',$$

we have a central extension C_z if G' by G'' with underlying k-scheme $G'' \times_k G'$ and product given by

$$(g'', g')(h'', h') = (g''h''z(g', h'), g'h').$$

The universal reductive k-group for the elliptic curve X is then

$$C_{u_X} \times_k SL_2.$$

This can be deduced from Theorem 1.1 using Atiyah's classification of vector bundles on X.

In general, an explicit determination of the k-group G_0 of (1.1) does not seem possible. It is however possible to describe appropriate quotients of G_0. In particular, under mild finiteness conditions, the quotient of G_0 by its semisimple part is an extension of the algebraic fundamental group of X by the protorus with group of characters the Picard group of the universal cover of X. When X is an elliptic curve for example, this quotient is C_{u_X}.

Even when the k-group G_0 is not known explicitly, its existence has strong consequences for principal bundles over X. The following theorem for example follows easily from (1.1).

Theorem 1.2. *Let X and k be as in Theorem* 1.1*, G be a reductive k-group, and P be a principal G-bundle. Then any two reductive k-subgroups of G which are minimal among reductive k-subgroups to which the structure group of P can be reduced are conjugate in G.*

Theorem 1.1 no longer holds if the hypothesis that k be algebraically closed is dropped, even if $X = \mathrm{Spec}(k)$. However if X has a k-point x, and we write $\widetilde{H}^1(X, x, G)$ for the subset of $H^1(X, G)$ consisting of the classes of those principal G-bundles that have a k-point above X, then the following modified form of Theorems 1.1 continues to hold.

Theorem 1.3. *Let X be a scheme over a field k of characteristic* 0 *for which $H^0(X, \mathcal{O}_X)$ is a henselian local ring. Then for any k-point x of X the functor $\widetilde{H}^1(X, x, -)$ on the category of reductive k-groups up to conjugacy is representable.*

For $X = \mathbf{P}^1$, the reductive k-group that represents the functor of Theorem 1.3 is again $\mathbf{G}_m$.

For X an elliptic curve, the reductive k-group that represents the functor of Theorem 1.3 is again $C_{u_X} \times_k SL_2$, provided that in the definition of C_{u_X} we replace $D(\mathrm{Pic}^0(X))$ by the k-group of multiplicative type $D(\mathrm{Pic}^0(X_{\overline{k}}))$ with Galois module of characters $\mathrm{Pic}^0(X_{\overline{k}})$, where $\overline{k}$ is an algebraic closure of k.

In the general case, where neither k is algebraically closed nor X has a k-point, it is possible to proceed by replacing principal bundles with groupoids. A groupoid over X is a k-scheme K with source and target morphisms d_1 and d_0 from K to X, together with a composition

$$K \times_{d_1 X^{d_0}} K \to K$$

which is associative and has identities and inverses. The target and source morphisms define on K a structure of scheme over $X \times_k X$, and the restriction of K to the diagonal is a group scheme K^{diag} over X. We have a category of groupoids over X up to conjugacy, where a morphism from K' to K is an equivalence class under conjugation by by cross-sections of K^{diag}. The groupoid K is said to be transitive affine if it is affine and faithfully flat over $X \times_k X$. Any principal G-bundle P over X gives rise to a transitive affine groupoid $\underline{\mathrm{Iso}}_G(P)$ over X, whose points in a k-scheme S above (x_0, x_1) are the isomorphisms from P_{x_1} to P_{x_0} of principal G-bundles over X_S. On the other hand the fibre above a k-point x of X of the source morphism of a transitive affine groupoid K is a principal bundle under the fibre of K^{diag} above x. A transitive affine groupoid K over X will be called reductive if the fibres of

K^{diag} are reductive. Then we have the following result, which can be shown to contain Theorems 1.1 and 1.3.

Theorem 1.4. *Let X be a scheme over a field k of characteristic 0 for which $H^0(X, \mathcal{O}_X)$ is a henselian local ring with residue field k. Then the category of reductive groupoids over X up to conjugacy has an initial object.*

For the formulation of general theorems, it is simple and convenient to work with groupoids over X as in Theorem 1.4, but this does not directly give a description of principal bundles over X. However an equivalent form of Theorem 1.4 analogous to Theorem 1.1 can be obtained by replacing the reductive k-groups of Theorem 1.1 by reductive groupoids over the spectrum of an algebraic closure of k.

Fix an algebraic closure $\overline{k}$ of k. Groupoids over $\mathrm{Spec}(\overline{k})$ will also be called groupoids over $\overline{k}/k$. A notion of principal bundle on a k-scheme X under a transitive affine groupoid F over $\overline{k}/k$ can be defined, and the isomorphism classes of such bundles form a set $H^1(X, F)$. If $G_{[\overline{k}]}$ is the constant groupoid over $\overline{k}$ defined by the affine k-group G, then

$$H^1(X, G) = H^1(X, G_{[\overline{k}]}).$$

In contrast to the case of affine k-groups, the set $H^1(X, F)$ for F a transitive affine groupoid over $\overline{k}/k$ may be empty. When X has a k-point, $H^1(X, F)$ is non-empty if and only if F is constant. The following is an equivalent version of Theorem 1.4.

Theorem 1.5. *Let X be as in Theorem 1.4 and $\overline{k}$ be an algebraic closure of k. Then the functor $H^1(X, -)$ on the category of reductive groupoids over $\overline{k}/k$ up to conjugacy is representable.*

Theorem 1.5 leads to a description of principal bundles under a reductive k-group over X, in the following way. We consider pairs (D, E) with D an affine $\overline{k}$-group and E an extension of topological groups

$$1 \to D(\overline{k})_{\overline{k}} \to E \to \mathrm{Gal}(\overline{k}/k) \to 1,$$

where the group $D(\overline{k})_{\overline{k}}$ of $\overline{k}$-points over $\overline{k}$ is given the pro-discrete topology defined by the quotients of D of finite type, and where the following condition is satisfied: conjugation of $D(\overline{k})_{\overline{k}}$ by an element of E above σ in $\mathrm{Gal}(\overline{k}/k)$ arises from an automorphism of the scheme D above the automorphism σ^{-1} of $\overline{k}$. Such pairs will be called Galois extended $\overline{k}$-groups. It will be shown that there is an equivalence from the category of transitive affine groupoids over $\overline{k}/k$ to the category of Galois extended $\overline{k}$-groups, which sends F to a pair (D, E) with $D = F^{\mathrm{diag}}$. In particular $G_{[\overline{k}]}$ is sent to

$$(G_{\overline{k}}, G(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k)).$$

For X as in Theorem 1.4, there then exists a pair (D_0, E_0) with D_0 reductive such that we have a bijection

$$\mathrm{Hom}((D_0, E_0), (G_{\overline{k}}, G(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k)))/G(\overline{k}) \xrightarrow{\sim} H^1(X, G) \tag{1.2}$$

which is natural in the reductive k-group G. If X is quasi-compact, the hypotheses of Theorem 1.1 are satisfied with X and k replaced by $X_{\overline{k}}$ and $\overline{k}$, and D_0 is the $\overline{k}$-group that represents the functor $H^1(X_{\overline{k}}, -)$ on reductive $\overline{k}$-groups up to conjugacy.

When $k = \overline{k}$ is algebraically closed, (1.2) reduces to (1.1). When $X = \mathrm{Spec}(k)$, the $\overline{k}$-group D_0 of (1.2) is trivial, and the set on the left of (1.2) is the set of splittings

of the extension $G(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k)$ modulo conjugation by $G(\overline{k})$, or equivalently, of continuous 1-cocycles of $\mathrm{Gal}(\overline{k}/k)$ with values in $G(\overline{k})$ modulo 1-coboundaries. In general, each k-point of X defines a splitting of the extension E_0 up to conjugation by $D_0(\overline{k})_{\overline{k}}$, so that E_0 is an obstruction to the existence of a k-point of X. The push forward of E_0 obtained by factoring out the identity component of D_0 is the the arithmetic fundamental group of X.

Let X be a curve over k of genus 0. Then (1.2) holds with $D_0 = \mathbf{G}_{m\overline{k}}$, and E_0 the extension of $\mathrm{Gal}(\overline{k}/k)$ by $\overline{k}^*$ given by the class of the Severi-Brauer variety X in the Brauer group $H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}^*)$ of k.

Let X be a curve over k of genus 1. Then X is a principal homogeneous space under its Jacobian X_0. We may identify the Galois modules $\mathrm{Pic}^0(X_{\overline{k}})$ and $\mathrm{Pic}^0(X_{0\overline{k}})$, because choosing a $\overline{k}$-point of X defines an isomorphism between them which is independent of the choice by translation invariance of Pic^0. Then we may form the k-group C_{u_X} as above with $D(\mathrm{Pic}^0(X))$ replaced by $D(\mathrm{Pic}^0(X_{\overline{k}}))$ and $\mathrm{Ker}\, n_X$ by $\mathrm{Ker}\, n_{X_0}$. For each $n \neq 0$ there is a short exact sequence of topological groups

$$1 \to C_{u_X}(\overline{k}) \xrightarrow{n_{u_X}} C_{u_X}(\overline{k}) \to X_0(\overline{k})_n \to 1 \tag{1.3}$$

with n_{u_X} the k-homomorphism defined by the power n^2 on the factor $D(\mathbf{Q})$ of C_{u_X} and the power n on the factor $D(\mathrm{Pic}^0(X_{\overline{k}}))$. Now for some n, the class of X in the Weil–Châtelet group $H^1(\mathrm{Gal}(\overline{k}/k), X_0(\overline{k}))$ of X_0 is the image of an element α_n of $H^1(\mathrm{Gal}(\overline{k}/k), X(\overline{k})_n)$. Then (1.2) holds with $D_0 = (C_{u_X})_{\overline{k}}$ and E_0 an extension whose class is given up to to an appropriate group of automorphisms of $C_{u_X}(\overline{k})$ by the image of α_n under the connecting map in a (non-commutative) exact cohomology sequence for (1.3).

The theorems above are proved by considering the category $\mathrm{Mod}(X)$ of vector bundles over X. It is a k-linear category with a tensor product. For G reductive, principal G-bundles over X may be identified with k-linear tensor functors from the category of representations of G to $\mathrm{Mod}(X)$. With the conditions on X above, $\mathrm{Mod}(X)$ has a unique maximal tensor ideal, and it can be shown that its quotient $\overline{\mathrm{Mod}(X)}$ by this ideal is a semisimple Tannakian category over k. The universal reductive k-group of Theorem 1.1 for example is then characterised by the property that its category of representations is equivalent as a Tannakian category to $\overline{\mathrm{Mod}(X)}$, and similarly for Theorems 1.3, 1.4 and 1.5.

The fundamental tool for proving the required universal properties is the splitting theorem for tensor categories of André and Kahn [AK02](see also [O'S05]), or more accurately a slightly more general form of this result proved below. It will follow from this result that the projection from $\mathrm{Mod}(X)$ to $\overline{\mathrm{Mod}(X)}$ has a splitting. This splitting gives a fibre functor from $\overline{\mathrm{Mod}(X)}$ with values in X, and from this we obtain the required universal objects.

The version of the above theorems proved below is in fact a rather more general one. We start with a groupoid H (not necessarily transitive affine) over X or more generally what will be called a pregroupoid over X, and consider principal bundles over X with an action of H, or transitive affine groupoids over X with a morphism from H. For appropriate H, we then obtain a classification of equivariant principal bundles, or of principal bundles equipped with a connection. In this more general form, Theorem 1.4 contains the generalised Jacobson–Morosov theorem of [AK02] and [O'S10], by taking $X = \mathrm{Spec}(k)$ and for H an affine k-group.

The paper is organised as follows. Sections 2 to 6 give the basic definitions for groupoids, pregroupoids and principal bundles. It is technically convenient here to work in the category of schemes over a base. In Sections 7 to 11, we specialise to the case where the base is a field k. From Section 12 on, we assume that k has characteristic 0. In Sections 12 and 13, the transitive affine groupoids that contain a reductive sungroupoid are characterised. The splitting theorem, in the form required, is proved in Section 14. The main theorem, the existence of universal reductive groupoids, is proved along with its corollaries in Section 15. These are then applied to principal bundles in Section 16, and further applications are given in Sections 17 and 18. Finally an explicit description of principal bundles under a reductive group over curves of genus 0 or 1 is given in Sections 19 and 20.

2. Groupoids and pregroupoids

This section contains the basic definitions and terminology for groupoids and pregroupoids, which will be used throughout the paper. For this foundational material, it is convenient to work initially in an arbitrary category $\mathcal{C}$ with fibre products, and to specialise later to the case where $\mathcal{C}$ is the category of schemes over a base.

Write $\widehat{\Delta}_2$ for the category with objects the ordered sets $[0] = \{0\}$, $[1] = \{0,1\}$ and $[2] = \{0,1,2\}$, with morphisms the injective order-preserving maps between them. A contravariant functor from $\widehat{\Delta}_2$ to a category $\mathcal{C}$ will be called a *pregroupoid in* $\mathcal{C}$, and a natural transformation of such functors a *morphism of pregroupoids in* $\mathcal{C}$. The image of an object or morphism in $\widehat{\Delta}_2$ under a pregroupoid will usually be written as a subscript. A pregroupoid H in $\mathcal{C}$ thus consists of objects $H_{[0]}$, $H_{[1]}$, $H_{[2]}$ in $\mathcal{C}$ and morphisms

$$H_{[0]} \leftleftarrows H_{[1]} \overset{\textstyle\leftarrow}{\leftleftarrows} H_{[2]}$$

satisfying the evident compatibilities. We write

$$d_i : H_{[n]} \to H_{[n-1]}$$

for H_{δ_i} with $\delta_i : [n-1] \to [n]$ the map whose image omits i, and occasionally

$$e_i : H_{[n]} \to H_{[0]}$$

for H_{ε_i} with $\varepsilon_i : [0] \to [n]$ the map whose image is i. For $n = 1$ we have $e_0 = d_1$ and $e_1 = d_0$. To every pregroupoid H there is associated a pregroupoid

$$H^{\mathrm{op}}$$

for which $(H^{\mathrm{op}})_{[n]}$ coincides with $H_{[n]}$ and $d_i : (H^{\mathrm{op}})_{[n]} \to (H^{\mathrm{op}})_{[n-1]}$ with $d_{n-i} : H_{[n]} \to H_{[n-1]}$.

Recall that a groupoid is a category in which every morphism is an isomorphism. We may regard a groupoid K as a set $K_{[0]}$ of objects, a set $K_{[1]}$ of arrows, a source map d_1 from $K_{[1]}$ to $K_{[0]}$, a target map d_0 from $K_{[1]}$ to $K_{[0]}$, and a composition map $\circ$ from the set of composable arrows

$$K_{[1]} \times_{d_1 K_{[0]} d_0} K_{[1]} \tag{2.1}$$

to $K_{[1]}$, such that the composition is associative and left and right identities and inverses exist for it (necessarily uniquely). A morphism of groupoids is then a functor between their underlying categories. Explicitly, a morphism from K to K' consists of maps $K_{[0]} \to K'_{[0]}$ and $K_{[1]} \to K'_{[1]}$ which commute with the source,

target and composition maps. Such a morphism necessarily preserves identities and inverses.

Let $\mathcal{C}$ be a category with fibre products. A *groupoid* K *in* $\mathcal{C}$ consists of objects $K_{[0]}$ and $K_{[1]}$ of objects and arrows in $\mathcal{C}$, source and target morphisms d_1 and d_0 from $K_{[1]}$ to $K_{[0]}$, and a morphism $\circ$ from the fibre product (2.1) to $K_{[1]}$, such that the points in any object of $\mathcal{C}$ form a groupoid. A morphism $K \to K'$ of groupoids in $\mathcal{C}$ is a pair of morphisms $K_{[0]} \to K'_{[0]}$ and $K_{[1]} \to K'_{[1]}$ which define on points of any object of $\mathcal{C}$ a morphism of groupoids, or equivalently which commute with the morphisms d_0, d_1, and $\circ$.

Let X be an object of $\mathcal{C}$. By a *groupoid over* X we mean a groupoid K in $\mathcal{C}$ with $K_{[0]} = X$. A morphism $K' \to K$ of groupoids over X is a morphism of groupoids from K' to K for which $K'_{[0]} \to K_{[0]}$ is the identity 1_X. The category of groupoids over X is thus a subcategory, in general non-full, of the category of groupoids in $\mathcal{C}$. A groupoid over X will usually be denoted by its object of arrows.

Any groupoid K in $\mathcal{C}$ may be regarded as a pregroupoid in $\mathcal{C}$ by taking for $K_{[2]}$ the object (2.1), with the morphisms d_0, d_1 and d_2 from $K_{[2]}$ to $K_{[1]}$ respectively the first projection, the composition and the second projection. A morphism $K \to K'$ of groupoids in $\mathcal{C}$ then extends uniquely to a morphism of pregroupoids in $\mathcal{C}$, with $K_{[2]} \to K'_{[2]}$ the fibre product of morphisms $K_{[1]} \to K'_{[1]}$ over $K_{[0]} \to K'_{[0]}$. Thus we obtain a fully faithful functor from groupoids in $\mathcal{C}$ to pregroupoids in $\mathcal{C}$. Any pregroupoid in the essential image of this functor will also be called a groupoid, so that the category of groupoids in $\mathcal{C}$ may be regarded as a full subcategory of that of pregroupoids in $\mathcal{C}$. A morphism from a pregroupoid to a groupoid in $\mathcal{C}$ is completely determined by its components at [0] and [1].

Let X be an object of $\mathcal{C}$. By a *pregroupoid over* X we mean a pregroupoid H in $\mathcal{C}$ with $H_{[0]} = X$. A morphism $H' \to H$ of pregroupoids over X is a morphism of pregroupoids from H' to H for which $H'_{[0]} \to H_{[0]}$ is the identity 1_X. The category of pregroupoids over X is thus a subcategory, in general non-full, of the category of pregroupoids in $\mathcal{C}$. It contains as a full subcategory the category of pregroupoids over X. If $\mathcal{C}$ has an initial object, then the category of pregroupoids over X has an initial object, with $H_{[1]}$ and $H_{[2]}$ initial. We write it simply as X. If $\mathcal{C}$ has a final object, then the category of prehroupoids over X has a final object, with $H_{[n]} = X^n$ and the d_i given by projections. It is a groupoid which we write as $[X]$.

Recall that a fibred category $\mathcal{F}$ (in the sense of [Gro62]) over $\mathcal{C}$ is the assignment to every object X of $\mathcal{C}$ of a category $\mathcal{F}_X$, the fibre above X, to every morphism $f : X' \to X$ of a pullback functor f^* from $\mathcal{F}_X$ to $\mathcal{F}_{X'}$, and to every composable pair of morphisms f' and f of a pullback isomorphism from $(f \circ f')^*$ to $f'^* f^*$, satisfying the following conditions: if f is the identity then f^* is the identity; if either f' or f is the identity then the associated pullback isomorphism is the identity; if f'', f' and f are composable, then the two isomorphisms from $(f \circ f' \circ f'')^*$ to $f''^* f'^* f^*$ defined using the pullback isomorphisms coincide ([Gro62, 190-02]).

As an example, we have the fibred category over $\mathcal{C}$ of relative objects, with fibre above X the category $\mathcal{C}/X$ of objects in $\mathcal{C}$ over X, pullback functors given by fibre product and pullback isomorphisms by uniqueness up to unique isomorphism of fibre products. It will always be assumed in what follows that the fibre products $X \times_X X'$ and $X' \times_X X$ coincide with X'. This ensures that the identity conditions for a fibred category are satisfied in this example. If $\mathcal{C}$ is the category of ringed spaces, we may take for the fibre above X the category of $\mathcal{O}_X$-modules, with pullback functors

given by the usual inverse image and pullback isomorphisms by uniqueness up to unique isomorphism of left adjoints.

H be a pregroupoid over X in $\mathcal{C}$. Given s fibred category $\mathcal{F}$ over $\mathcal{C}$ and an object Z in $\mathcal{F}_X = \mathcal{F}_{H_{[0]}}$, an *action of H on Z* is an isomorphism

$$\alpha : {d_1}^* Z \xrightarrow{\sim} {d_0}^* Z$$

in $\mathcal{F}_{H_{[1]}}$ such that if for $i = 0, 1, 2$ we write α_i for the morphism

$$(d_1 \circ d_i)^* Z \xrightarrow{\sim} d_i^* {d_1}^* Z \xrightarrow{{d_i}^* \alpha} d_i^* {d_0}^* Z \xrightarrow{\sim} (d_0 \circ d_i)^* Z$$

in $\mathcal{F}_{H_{[2]}}$ defined using the pullback isomorphisms, then

$$\alpha_1 = \alpha_0 \circ \alpha_2.$$

Given also Z' in $\mathcal{F}_X$ with an action α' of H, a morphism from Z' to Z is a morphism $f : Z' \to Z$ in $\mathcal{F}_X$ such that $\alpha \circ {d_1}^* f = {d_0}^* f \circ \alpha'$. If $\mathcal{C}$ has an initial object and H is the initial pregroupoid X over X, then the category of objects in $\mathcal{F}_X$ with an action of H may be identified with $\mathcal{F}_X$.

When $\mathcal{F}$ is the fibred category over $\mathcal{C}$ of relative objects, an object in $\mathcal{F}_X$ with an action of H will also be called an H-object.

An object J in $\mathcal{C}$ equipped with a source morphism d_1 and a target morphism d_0 from J to X will be called a graph over X. Thus for example $H_{[1]}$ is a graph over X. For any graph J over X, it will always be assumed, unless the contrary is stated, that

$$J \times_X X' = J \times_{d_1 X} X'$$

for J on the left of a fibre product over X, and

$$X' \times_X J = X' \times_{X^{d_0}} J$$

for J on the right of a fibre product. In particular this will be assumed for $H_{[1]}$.

An H-object is the same as an an object X' over X in $\mathcal{C}$ and a morphism

$$\widetilde{\alpha} : H_{[1]} \times_X X' \to X'$$

over X, where $H_{[1]} \times_X X'$ is regarded as a scheme over X by composing the projection onto $H_{[1]}$ with d_0, such that the morphism

$$H_{[1]} \times_X X' \to X' \times_X H_{[1]}$$

over $H_{[1]}$ defined by $\widetilde{\alpha}$ is an isomorphism, and such that the diagram

$$\begin{tikzcd}
H_{[2]} \times_{e_0 X} X' \arrow[r, "d_1 \times_X X'"] \arrow[d, "(d_0, d_2) \times_X X'"'] & H_{[1]} \times_X X' \arrow[r, "\widetilde{\alpha}"] & X' \arrow[d, equal] \\
H_{[1]} \times_X H_{[1]} \times_X X' \arrow[r, "H_{[1]} \times_X \widetilde{\alpha}"] & H_{[1]} \times_X X' \arrow[r, "\widetilde{\alpha}"] & X'
\end{tikzcd}$$

commutes.

Let H be a pregroupoid over X and X' be an H-object. Then with $H_{[2]}$ regarded as a scheme over X using e_0, there is a unique pregroupoid H' over X' with

$$(H')_{[n]} = H_{[n]} \times_X X'$$

for $n = 0, 1, 2$, such that

(1) the projections $(H')_{[n]} \to H_{[n]}$ define a morphism $H' \to H$,
(2) $e_0 : (H')_{[n]} \to X'$ is the projection for each n,
(3) $d_0 : (H')_{[1]} \to X'$ is the action of H on X'.

We write this pregroupoid H' as

$$H \times_X X'.$$

It should be noted however that $H \times_X X'$ is not the fibre product of pregroupoids, and that even when $\mathcal{C}$ has an initial object, there is in general no morphism from H to the initial pregroupoid X over X. If H is a groupoid then $H \times_X X'$ is a groupoid. There is an evidently defined functor $H \times_X -$ from H-objects in $\mathcal{C}$ to pregroupoids in $\mathcal{C}$ equipped with a morphism to H. It is fully faithful, with essential image those $\widetilde{H} \to H$ such that the square

$$\begin{array}{ccc} \widetilde{H}_{[m]} & \xrightarrow{\widetilde{H}_\varphi} & \widetilde{H}_{[n]} \\ \downarrow & & \downarrow \\ H_{[m]} & \xrightarrow{H_\varphi} & H_{[n]} \end{array}$$

is cartesian for every $\varphi : [n] \to [m]$ in $\widehat{\Delta}_2$.

Suppose that $\mathcal{C}$ has a final object. We define the pullback of a pregroupoid H over X along $X' \to X$ as the fibre product

$$H \times_{[X]} [X']$$

in the category of pregroupoids in $\mathcal{C}$. If H is a groupoid then $H \times_{[X]} [X']$ is a groupoid. If X' has a structure of H-object, there is a canonical morphism

$$H \times_X X' \to H \times_{[X]} [X']$$

of pregroupoids over X', with components the canonical morphism of pregroupoids from $H \times_X X'$ to H and the unique morphism of pregroupoids over X' from $H \times_X X'$ to $[X']$.

Suppose now that $\mathcal{C}$ is the category of schemes over a scheme S. Let H be a pregroupoid over X in $\mathcal{C}$. Then an H-object will also be called an *H-scheme*. By an *H-subscheme* of an H-scheme X' we mean a subscheme X'' of X such that the action of H on X' restricts to an action of H on X''. Such an X'' has a unique structure of H-scheme for which embedding is a morphism of H-schemes. An H-subscheme of X' is the same as an $(H \times_X X')$-subscheme of the final $H \times_X X'$-scheme X'.

Let K be a groupoid over X. By a *subgroupoid of* K we mean a subscheme K' of K such the composite of any two composable points of K' lies in K', the identities of K lie in K', and the inverse of any point of K' lies in K'. Such a K' has a unique structure of groupoid over X for which the embedding is a morphism of groupoids over X. The inverse image of the diagonal X of $X \times_S X$ under the morphism (d_0, d_1) from K to $X \times_S X$ is a subgroupoid

$$K^{\mathrm{diag}}$$

of K, and the composition defines on it a stucture of group scheme over X.

Let X be a scheme. We write

$$\mathrm{MOD}(X)$$

for the category of quasi-coherent $\mathcal{O}_X$-modules. Locally free $\mathcal{O}_X$-modules of finite type will also be called vector bundles over X. We write

$$\mathrm{Mod}(X)$$

for the full subcategory of $\mathrm{MOD}(X)$ of vector bundles over X. Then we have fibred categories MOD and Mod over the category of schemes over S with respective fibres $\mathrm{MOD}(X)$ and $\mathrm{Mod}(X)$ above X, and with pullback functors and pullback isomorphisms the same as those for ringed spaces above. Given a pregroupoid H over X, we write

$$\mathrm{MOD}_H(X)$$

for the category of H-objects in MOD. An object of $\mathrm{MOD}_H(X)$ will also be called an *H-module*. We write

$$\mathrm{Mod}_H(X)$$

for the category of H-objects in Mod. An object in $\mathrm{Mod}_H(X)$ will also be called a *representation of H*.

We may regard $\mathcal{O}_X$ as a representation of H, with the action of H, modulo the canonical isomorphism from $\mathcal{O}_{H_{[1]}}$ to $d_i^*\mathcal{O}_X$, the identity. For every H-module $\mathcal{V}$, we write

$$H^0_H(X, \mathcal{V})$$

for the subgroup of $H^0(X, \mathcal{V})$ consisting of those sections s for which the action of H sends ${d_1}^*s$ to ${d_0}^*s$. It is an $H^0_H(X, \mathcal{O}_X)$-submodule of $H^0(X, \mathcal{V})$, which may be identified with the group of homomorphisms of H-modules from $\mathcal{O}_X$ to $\mathcal{V}$.

3. Transitive groupoids

This section contains the basic definitions and terminolonology for the *fpqc* topology and for transitive affine groupoids, which are closely related to principal bundles. It is technically convenient to work here in the category of schemes over a arbitary base, although for the applications later the base will be the spectrum of a field.

A morphism of of schemes $X' \to X$ will be called an *fpqc covering morphism* if every point of X is contained in an open subscheme U of X such that there exists a faithfully flat quasi-compact morphism $U' \to U$ which factors through the restriction of $X' \to X$ to the inverse image of U.

The property of being *fpqc* covering is local on the target. Any faithfully flat quasi-compact morphism is *fpqc* covering. If X' is the disjoint union of a family of open subschemes covering X, then $X' \to X$ with components the embeddings is *fpqc* covering. The composite of two *fpqc* covering morphisms is *fpqc* covering. If $Z' \to Z$ is the pullback of $X' \to X$ along $Z \to X$, then $Z' \to Z$ is *fpqc* covering if $X' \to X$ is, and the converse holds when $Z \to X$ is *fpqc* covering. If $X' \to X$ factors through an *fpqc* covering morphism $X'' \to X$, then $X' \to X$ is *fpqc* covering.

Let $X' \to X$ be an *fpqc* covering morphism. Then the pullback functors induced by $X' \to X$ from quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_{X'}$-modules, and from schemes over X to schemes over X', are both faithful. A quasi-coherent $\mathcal{O}_X$ module is locally free of finite type if and only its pullback onto X' is locally free of finite type, and a scheme over X is affine if and only if its pullback onto X' is affine.

An $[X]$-module is a quasi-coherent $\mathcal{O}_S$-module equipped with a descent datum from X to S, and an $[X]$-scheme is a scheme over X equipped with a descent datum from X to S. We have faithfully flat descent for modules and for affine schemes and their morphisms: if $X \to S$ is *fpqc* covering, then pullback along $X \to S$ induces an equivalence from quasi-coherent $\mathcal{O}_S$-modules to $[X]$-modules, and from schemes affine over S to affine $[X]$-schemes. We also have faithfully flat descent for

morphisms of schemes: if $X \to S$ is *fpqc* covering, then pullback along $[X] \to S$ induces a fully faithful functor from schemes over S to $[X]$-schemes. These descent properties are contained in Lemma 3.2 below as the case where $H = X = S$.

For the proof of Lemma 3.2 and elsewhere, the following fact about pullbacks of groupoids will be useful. Let K be a groupoid over X and h be a point of K in a scheme Z over S. If we write K_i for the pullback of K along $d_i(h) : Z \to X$, then conjugation with h defines an isomorphism

$$\varphi_h : K_1 \xrightarrow{\sim} K_0 \tag{3.1}$$

of groupoids over Z. For $\mathcal{V}$ a K-module, the action of h on $\mathcal{V}$ then defines an isomorphism of K_1-modules

$$\mathcal{V}_{d_1(h)} \xrightarrow{\sim} {\varphi_h}^* \mathcal{V}_{d_0(h)} \tag{3.2}$$

which is natural in $\mathcal{V}$. There is is similar natural isomorphism for K-schemes.

A groupoid K over X/S will be called *transitive* if both the structural morphism of X and

$$(d_0, d_1) : K \to X \times_S X$$

are *fpqc* covering morphisms.

Lemma 3.1. *Let K be a transitive groupoid over X/S and $a : X' \to X$ be a morphism with $X' \to S$ fpqc covering. Then there exist an fpqc covering morphism $b : Z \to X$, a retraction $b' : Z \to X'$, and an isomorphism φ of groupoids over Z from the pullback K_1 of K along $a \circ b'$ to the pullback K_0 of K along b, with the following property: the pullback functor along $a \circ b'$ from K-schemes (resp. K-modules) to K_1-schemes (resp. K_1-modules) is naturally isomorphic to the pullback functor functor along b composed with restriction along $\varphi : K_1 \xrightarrow{\sim} K_0$.*

Proof. Take for Z the fibre product of K with $X \times_S X'$ over $X \times_S X$. If $h : Z \to K$ and $(b, b') : Z \to X \times_S X'$ are the projections, then b is *fpqc* covering and b' has a right inverse whose component at K is a composed with the identity of K and whose component at $X \times_S X'$ is $(a, 1_{X'})$. Also $d_0(h) = b$ and $d_1(h) = b' \circ a$. If φ is the isomorphism φ_h of (3.1), then (3.2) gives the required natural isomorphism. □

Lemma 3.2. *Let H be a pregroupoid over X and H' be the pullback of H along $X' \to X$. Suppose either that $X' \to X$ is fpqc covering or that there is a scheme S and a structure on X of scheme over S such that $X' \to S$ is fpqc covering and H is a transitive groupoid over X/S. Then pullback along $X' \to X$ induces an equivalence from H-modules (resp. representations of H, resp. affine H-schemes) to H'-modules (resp. representations of H', resp. affine H'-schemes), and a fully faithful functor from H-schemes to H'-schemes.*

Proof. We need only consider the case where $X' \to X$ is *fpqc* covering: the other case follows from this one by applying it to the morphisms b and b' of Lemma 3.1.

Consider the following three types of morphism:

(A) a faithfully flat quasi-compact morphism,
(B) a surjective morphism whose source is a disjoint union of open subschemes U of its target, with restriction to each U the embedding,
(C) a retraction.

Since $X' \to X$ is *fpqc* covering, there is a composite $X'' \to X$ of a morphism of type (A) with one of type (B) which factors through $X' \to X$. Taking the fibre product

of X' and X'' over X we obtain a commutative square with one side $X' \to X$, two sides the composite of a morphism of type (A) with one of type (B), and one side of type (C). Thus we may suppose that $X' \to X$ is of type (A), (B) or (C).

Suppose first that $H = X$, so that $H' = [X']_X$ and we are to prove that faithfully flat descent holds for $X' \to X$. In case (A), this follows from [SGA71, VIII 1.1, 2.1 and 5.2]. In case (B), it follows from gluing of modules or schemes along open subschemes. For (C), note that if $e : X' \to X'$ is the composite of $X' \to X$ with a right inverse, then by (3.2) with X'/X for X/S, X' for Z, $[X']_X$ for K, and $(1_{X'}, e)$ for h, the endofunctor on $[X']_X$-modules or of $[X']_X$-schemes defined by pullback along e is isomorphic to the identity.

For arbitrary H, we have a commutative square of groupoids

(3.3)
$$\begin{array}{ccc} [X']_X & \longrightarrow & H' \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & H \end{array}$$

defined by pullback along $X \to H$. We show that pullback induces an equivalence from H-modules to H'-modules. The cases of representations of H and of H-schemes are similar.

Let $\mathcal{V}$ and $\mathcal{W}$ be H-modules. Since $H'_{[1]} \to H_{[1]}$ is $fpqc$ covering, any morphism of $\mathcal{O}_X$-modules $\mathcal{V} \to \mathcal{W}$ whose pullback along $X' \to X$ is a morphism of H'-modules is a morphism of H-modules. The required full faithfulness thus follows from (3.3) and the case where $H = X$.

By the case where $H = X$, the restriction of any H'-module to $[X']_X$ is isomorphic to pullback along $X' \to X$ of a quasi-coherent $\mathcal{O}_X$-module. For the required essential surjectivity, it is thus enough to prove the following: given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{V}$ with pullback $\mathcal{V}_{X'}$ onto X', any structure

$$\alpha' : {d_1}^*\mathcal{V}_{X'} \xrightarrow{\sim} {d_0}^*\mathcal{V}_{X'}$$

of H'-module on $\mathcal{V}_{X'}$ with underlying $[X']_X$-module the pullback of $\mathcal{V}$ along $X' \to X$ is the pullback along $H'_{[1]} \to H_{[1]}$ of an isomophism of $\mathcal{O}_{H_{[1]}}$-modules

$$\alpha : {d_1}^*\mathcal{V} \xrightarrow{\sim} {d_0}^*\mathcal{V}.$$

Indeed since $H'_{[2]} \to H_{[2]}$ is $fpqc$ covering, the required compatibility for α will follow from that for α'.

Let h be a point of $H_{[1]}$ in some scheme, and

$$h' = (h, x'_0, x'_1)$$

be a point of $H'_{[1]}$ lying above h. Modulo the canonical pullback isomorphisms, the pullback of α' along h' is an isomorphism

$$\alpha'_{h'} : \mathcal{V}_{d_1(h)} \xrightarrow{\sim} \mathcal{V}_{d_0(h)}.$$

Since $X' \to X$ is of type (A), (B) or (C), so also is $H'_{[1]} \to H_{[1]}$. By the case where $H = X$ with $H_{[1]}$ for X and $H'_{[1]}$ for X', the required α will thus exist provided that α' is a morphism of $[H'_{[1]}]_{H_{[1]}}$-modules, i.e. provided that $\alpha'_{h'}$ is independent of h' above h for any given h.

If x' and y' are points of X' lying above the point x of X, then $\alpha'_{(s_0(x),x',y')}$ is the identity, because the restriction of α' to $[X']_X$ is the pullback isomorphism.

Also if j' is a point of $H'_{[1]}$, then

$$\alpha'_{d_0(j')} \circ \alpha'_{d_2(j')} = \alpha'_{d_1(j')},$$

by the compatibility condition for α'. Taking $(s_0(h), x'_0, y'_0, x'_1)$ for j' shows that $\alpha'_{h'}$ does not depend on x'_0, and taking $(s_1(h), x'_0, x'_1, y'_1)$ for j' shows that it does not depend on x'_1. □

Let T be a contravariant functor T from schemes over S to sets. We say that T has the *descent property* if for every *fpqc* covering morphism $Z' \to Z$ of schemes over S, the diagram

$$T(Z) \longrightarrow T(Z') \rightrightarrows T(Z' \times_Z Z')$$

defined by $Z' \to Z$ and the projections from $Z' \times_Z Z'$ is an equaliser diagram. If T is representable, then it has the descent property: apply the full faithfulness of Lemma 3.2 with $X = S = Z$ and $H = [Z']$ to $\mathrm{Hom}_Z(Z, W_Z)$ for W over S.

Given a contravariant functor T from schemes over S to sets and a scheme S' over S, write $T_{/S'}$ for the functor on schemes over S' with

$$T_{/S'}(Z') = T(Z')$$

Similarly, given a natural transformation $\xi : T' \to T$, write $\xi_{/S'}$ for the natural transformation $T'_{/S'} \to T_{/S'}$ with

$$(\xi_{/S'})_{Z'} = \xi_{Z'}.$$

We have $T_{/S} = T$, and $(T_{/S'})_{/S''} = T_{/S''}$ for a scheme S'' over S'. If T' is the functor represented by Z over S, then $T_{/S'}$ may be identified with the functor represented by $Z_{S'}$ over S', and with this identification, if $\xi : T' \to T$ corresponds to z in $T(Z)$, then $\xi_{/S'}$ corresponds to z' in $T_{/S'}(X_{S'}) = T(X_{S'})$ where z' is the image of z under T applied to the projection $Z_{S'} \to Z$. If T and T' have the descent property, $S' \to S$ is *fpqc* covering, and $\xi_{/S'} : T'_{/S'} \to T_{/S'}$ is an isomorphism, then $\xi : T' \to T$ is an isomorphism: for any Z over S, the projection from $Z_{S'}$ gives an *fpqc* covering morphism $Z' \to Z$ over S with Z' a scheme over S', so that ξ_Z is bijective because $\xi_{Z'}$ and $\xi_{Z' \times_Z Z'}$ are.

Lemma 3.3. *Let S' be a scheme over S and T be a contravariant functor from schemes over S to sets. Suppose that $S' \to S$ is fpqc covering, that T has the descent property, and that $T_{/S'}$ is representable by a scheme affine over S'. Then T is representable by a scheme affine over S.*

Proof. Let Z' be a scheme affine over S' which represents $T_{/S'}$, and z' in $T_{/S'}(Z') = T(Z')$ be the universal element. Write Z'_0 and Z'_1 for the pullbacks of Z' along the projections $S' \times_S S' \to S$, and z'_i for the image in $T_{/(S' \times_S S')}(Z'_i) = T(Z'_i)$ of z' under T applied to $Z'_i \to Z'$. Then Z'_i represents $T_{/(S' \times_S S')}$ with universal element z'_i. Thus there is a unique isomorphism

$$\alpha : Z'_1 \xrightarrow{\sim} Z'_0$$

over $S' \times_S S'$ such that $T(\alpha)$ sends z'_0 to z'_1. Further α satisfies the compatibility condition required for a structure of $[S']$-scheme on Z'.

By Lemma 3.2, we have an $[S']$-isomorphism $u : Z_{S'} \xrightarrow{\sim} Z'$ for some scheme Z affine over S. Then $Z_{S'}$ represents $T_{/S'}$ with universal element $T(u)(z')$. The images of $T(u)(z')$ under T applied to the projections

$$Z_{S'} \times_Z Z_{S'} \xrightarrow{\sim} Z_{S' \times_S S'} \to Z_{S'}$$

coincide, because $T(\alpha)$ sends z'_0 to z'_1. Since T has the descent property, $T(u)(z')$ is thus the image under T applied to $Z_{S'} \to Z$ of a (unique) z in $T(Z)$. Then Z represents T with universal element z, because if ξ is the natural transformation that corresponds to z, then $\xi_{/S'}$ corresponds to $T(u)(z')$. $\square$

Let K be a groupoid over X/S. If X' is a K-scheme and $s : X \to X'$ is a cross-section of X' over X, then the points v of K such that v sends $s(d_1(v))$ to $s(d_0(v))$ are those of a subgroupoid of K, which may be identified with the pullback

$$(K \times_X X')_{[X']}[X]$$

of $K \times_X X'$ along s. We call this subgroupoid the *stabiliser* of s.

A K-scheme X' will be called *transitive* if the groupoid $K \times_X X'$ over X'/S is transitive. The stabiliser of any cross-section of a transitive K-scheme is a transitive subgroupoid of K.

Let K be a transitive affine groupoid over X/S and K' be a transitive affine subgroupoid of K over X/S. Then a pair (Y, y) consisting of a transitive K-scheme Y and a section y of Y over X with stabiliser K' will be called a quotient of K by K'. If Z is a K-scheme and z is section of Z over X which is stabilised by K', then we have a commutative square

$$\begin{tikzcd}
K \times_X K' \arrow[r] \arrow[d] & K \arrow[d, "p"] \\
K \arrow[r, "p"] & Z
\end{tikzcd}$$

with p the restriction of the action of K on Z to the subscheme K of $K \times_X Z$ defined by z, and the top and left arrows the composition and the first projection. When $(Z, z) = (Y, y)$, the square is cartesian because K' is the stabiliser of y, and p is is an *fpqc* covering morphism because K acts transitively on Y, so that p is the coequaliser of the top and left arrows. Thus (Y, y) is initial in the category of pairs (Z, z). In particular (Y, y) is unique up to unique isomorphism. We write K/K' for Y and call y the base section of K/K'. When $X = S$, the notion of quotient K/K' coincides with the usual one of group schemes over S.

In general, the quotient K/K' need not exist, even if $X = S$ and S is the spectrum of a field. When K/K' exists, restricting the action from K to K^{diag} shows that the quotient $K^{\mathrm{diag}}/K'^{\mathrm{diag}}$ of group schemes exists and coincides with K/K'. Conversely when $K^{\mathrm{diag}}/K'^{\mathrm{diag}}$ exists, the action on it of K^{diag} extends uniquely to an action of K, and $K^{\mathrm{diag}}/K'^{\mathrm{diag}}$ is then the quotient K/K'. This can be seen by reducing first, using faithfully flat descent of morphisms and appropriate base change and pullback, to the case where K' and hence K is constant and X/S has a cross section. In particular, if X is the spectrum of a field and K is of finite type, then K/K' always exists.

The quotient K/K' is preserved when it exists by base extension and pullback. Conversely if the quotient of either the base extensions along an *fpqc* covering morphism or the pullbacks along some morphism exists and is affine, then K/K' exists and is affine. In particular, if X is the spectrum of a field k and K is of finite type, it follows in the usual way that if the quotient of the fibre of K by the fibre of K' at some point in an extension of k of the diagonal is affine, then K/K' exists and is affine. This holds in particular if k has characteristic 0 and K' is reductive.

4. Torsors

This section and the two following contain the foundational material that will be required on torsors and principal bundles. The notion of a torsor under an affine group scheme is an absolute one. When working over a base scheme, principal bundles are the torsors under constant affine group schemes.

Let X be a scheme and J be an affine group scheme over X. A right action of J on a scheme P over X is a morphism

$$P \times_X J \to P \tag{4.1}$$

over X which satisfies the usual unit and associativity conditions. A scheme equipped with a right action J will be called a *right J-scheme*. We have a category of right J-schemes, whose morphisms are those morphisms of schemes over X which are compatible with the action of J. The composition of J defines a structure of right J-scheme on J. A right J-scheme will be called *trivial* if it is isomorphic to J. Pullback along a morphism $X' \to X$ defines a functor $-_{X'}$ from right J-schemes to right $J_{X'}$-schemes. A morphism $J_{X'} \to P_{X'}$ of right $J_{X'}$-schemes may be identified with a section of P in X' over X. Taking in particular $X' = P$, corresponding to the identity of P there is a canonical morphism

$$P \times_X J \to P \times_X P \tag{4.2}$$

of right J_P-schemes, with first component the first projection and second component the action (4.1).

Let $j : J \to J'$ is a morphism of affine group schemes over X. Then we have a restriction functor along j from right J'-schemes to right J-schemes. If P is a J-scheme and P' is a J'-scheme, then a morphism $q : P \to P'$ of schemes over X will be said to be *compatible with* j if it is a morphism of J-schemes from P to the restriction of P' along j. It is equivalent to require that $q \times j$ composed with the action of J' on P' coincide with the the action of G on P composed with q. A J-morphism of right J-schemes is the same as a morphism of the underlying schemes over X compatible with 1_J.

A *J-torsor* is a right J-scheme P for which there exists an *fpqc* covering morphism $X' \to X$ such that the right $(J \times_X X')$-scheme $P \times_X X'$ is trivial. Any morphism of J-torsors is an isomorphism. A right J-scheme P is a J-torsor if an only if $P \to X$ is *fpqc* covering and (4.2) is an isomorphism. We write

$$H^1(X, J)$$

for the set of isomorphism classes of J-torsors. It is a pointed set with base point the class of J.

Lemma 4.1. *Let J be an affine group scheme over X and P be a J-torsor.*

(i) *If $u : J \to J'$ is a morphism of affine group schemes over X, then there exist a J'-torsor P' and a morphism $P \to P'$ over X compatible with u.*

(ii) *If $u_i : J \to J_i$ for $i = 1, 2$ and $v : J_1 \to J_2$ are morphisms of affine group schemes over X with $v \circ u_1 = u_2$, and if P_i is a J_i-torsor and $q_i : P \to P_i$ is compatible with u_i, then there is a unique morphism $q : P_1 \to P_2$ over X compatible with v such that $q \circ q_1 = q_2$.*

Proof. Let $X' \to X$ be an *fpqc* covering morphism along which the pullback of P is trivial. To prove (ii), we may suppose, after pulling back along $X' \to X$ and applying the full faithfulness of Lemma 3.2 with $H = X$, that $P = J$ is trivial. We

may then suppose further that $P_i = J_i$ is trivial and $q_i = u_i$, so that the unique q is v.

By (ii) with $u_1 = u_2$ and v the identity, the pair P' and $P \to P'$ of (i) is unique up to unique J'-isomorphism if it exists. To prove (i), we may thus suppose, after pulling back along $X' \to X$ and applying the essential surjectivity of Lemma 3.2 with $H = X$, that $P = J$ is trivial. We may then take the trivial J'-torsor J' for P' and u for $P \to P'$. □

Let P be a J-torsor and $u : J \to J'$ be a morphism of group schemes over X. Then by Lemma 4.1, a pair consisting of a J'-torsor P' and a morphism $P \to P'$ compatible with u exists, and is unique up to unique J'-isomorphism. We say that P' is the *push forward of P along u*. The push forward of P along an inner automorphism of J is P itself, because if j is a cross-section of J, then right translation $P \to P$ by j^{-1} is compatible with conjugation $J \to J$ by j.

We have a functor $H^0(X, -)$ from affine group schemes over X to groups, defined by taking cross-sections. By assigning to the morphism u the map defined by push forward along u, we also have a functor $H^1(X, -)$ from affine group schemes over X to pointed sets. It preserves finite products, so that $H^1(X, J)$ has a structure of abelian group when J is commutative. It also sends inner automorphisms of J to the identity of $H^1(X, J)$. Thus we may regard $H^1(X, -)$ as a functor on the category of affine group schemes over X up to conjugacy, where a morphism from J' to J is an orbit under the action by composition of the group of inner automorphisms of J on the set of morphisms $J' \to J$ of affine group schemes over X.

We say that morphisms $J'' \to J$ and $J \to J'$ of affine group schemes over X form short exact sequence

$$1 \to J'' \to J \to J' \to 1 \tag{4.3}$$

if $J \to J'$ is *fpqc* covering and $J'' \to J$ is a closed immersion with image the kernel of $J \to J'$. To (4.3) there is associated functorially an exact cohomology sequence

$$1 \to H^0(X, J'') \to H^0(X, J) \to H^0(X, J') \xrightarrow{\delta} H^1(X, J'') \to H^1(X, J) \to H^1(X, J')$$

of pointed sets, where the connecting map δ sends j' to the class of the inverse image of j' in J, and the other maps are defined by functoriality.

Let H be a pregroupoid over X, and suppose that J is equipped with a structure of H-group scheme. By an *(H, J)-scheme* we mean an H-scheme P together with a structure of right J-scheme on P for which the action (4.1) of J on P is an H-morphism. The condition that (4.1) be an H-morphism is equivalent to the condition that the isomorphism over $H_{[1]}$ defining the H-structure of P be compatible with the isomorphism over $H_{[1]}$ defining the H-structure of J. A morphism $P \to P'$ of (H, J)-schemes is a morphism of schemes over X which is compatible with the actions of H and J. We say that an (H, J)-scheme is an *(H, J)-torsor* if its underlying right J-scheme is an (H, J)-torsor. Any morphism of (H, J)-torsors is an isomorphism. We write

$$H^1_H(X, J)$$

for the set of isomorphism classes of (H, J)-torsors. It is a pointed set with base point the class of J.

Lemma 4.2. *Let P be an (H, J)-torsor, $u : J \to J'$ be a morphism of affine H-groups, P' be a J'-torsor, and $q : P \to P'$ be a morphism over X compatible*

with u. Then there is a unique structure of H-scheme on P' such that P' is an (H, J')-torsor and q is a morphism of H-schemes.

Proof. Apply Lemma 4.1(ii) to the pullbacks of P and P' onto the $H_{[i]}$. □

Let P be an (H, J)-torsor and $u : J \to J'$ be a morphism of affine H-groups. Then by Lemmas 4.1 and 4.2, a pair consisting of an (H, J')-torsor P' and an H-morphism $P \to P'$ compatible with u exists, and is unique up to unique (H, J')-isomorphism. We say that P' is the *push forward of P along u*. The inner automorphism of J defined by an H-invariant cross-section of J is an H-morphism, and the push forward of P along such an H-morphism is P itself.

We have a functor $H^0_H(X, -)$ from affine H-groups to groups, defined by taking H-invariant cross-sections. By assigning to the morphism u the map defined by push forward along u, we also have a functor $H^1_H(X, -)$ from affine H-groups to pointed sets. It preserves finite products, so that $H^1_H(X, J)$ has a structure of abelian group when J is commutative. It also sends H-inner automorphisms of J, i.e. those defined by an H-invariant cross-section of J, to the identity of $H^1(X, J)$. Thus we may regard $H^1(X, -)$ as a functor on the category of affine H-groups up to conjugacy, where a morphism from J' to J is an orbit under the action by composition of the group of H-inner automorphisms of J on the set of morphisms $J' \to J$ of affine group schemes over X.

We say that morphisms $J'' \to J$ and $J \to J'$ of affine H-groups form a short exact sequence if (4.3) is a short exact sequence of affine group schemes over X. To such a short exact sequence of affine H-groups there is associated functorially an exact cohomology sequence

$$1 \to H^0_H(X, J'') \to H^0_H(X, J) \to H^0_H(X, J') \xrightarrow{\delta} H^1_H(X, J'') \to H^1_H(X, J) \to H^1_H(X, J')$$

of pointed sets, where the connecting map δ sends j' to the class of the inverse image of j' in J, and the other maps are defined by functoriality.

Lemma 4.3. *Let H and X' be as in Lemma 3.2, and denote by H' and J' the respective pullbacks of H and J along $X' \to X$. Then pullback along $X' \to X$ induces an equivalence from (H, J)-torsors to (H', J')-torsors.*

Proof. By Lemma 3.2, pullback along $X' \to X$ induces an equivalence from affine (H, J)-schemes to affine (H', J')-schemes. Given an affine (H, J)-scheme P, it is thus enough to show that $P \to X$ is $fpqc$-covering if and only if $P_{X'} \to X'$ is, and that (4.2) is an isomorphism if and only if its pullback along $X' \to X$ is. This is clear in the case where $X' \to X$ is $fpqc$ covering, and in the other case it follows from Lemma 3.1, because (4.2) is a morphism of H-schemes. □

Lemma 4.4. *Let K be a transitive affine groupoid over X/S and $X' \to X$ be a morphism with $X' \to S$ $fpqc$ covering. Denote by K' the pullback of K along $X' \to X$. Then pullback along $X' \to X$ induces an equivalence from right K-schemes which are K^{diag}-torsors to right K'-schemes which are K'^{diag}-torsors.*

Proof. By Lemma 3.2, pullback along $X' \to X$ induces an equivalence from affine right K-schemes to affine right K'-schemes. Given an affine right K-scheme P, it is thus enough to show that $P \to X$ is $fpqc$-covering if and only if its pullback along $X' \to X$ is, and that (4.2) with $J = K^{\mathrm{diag}}$ is an isomorphism if and only if its pullback along $X' \to X$ is. This is clear from Lemma 3.1, because with the action by conjugation of K on $J = K^{\mathrm{diag}}$, (4.2) is a morphism of K-schemes. □

Lemma 4.5. *Let K be a transitive affine groupoid over X/S, and P_0 and P_1 be right K-schemes which are K^{diag}-torsors. Then the functor on schemes over S that sends Z to the set of K_Z-isomorphisms from P_{1Z} to P_{0Z} is representable by a scheme which is affine over S with structural morphism fpqc covering.*

Proof. Denote by T the functor on schemes over S that sends Z to the set K_Z-isomorphisms from P_{1Z} to P_{0Z}. If S' is a scheme over S, and if X', K', P'_0 and P'_1 are obtained from X, K, P_0 and P_1 by base change along $S' \to S$, then $T_{/S'}$ is the functor on schemes over S' that sends Z' to the set of $K'_{Z'}$-isomorphisms from $P'_{1Z'}$ to $P'_{0Z'}$. Since $Z_{S'}$ represents $T_{/S'}$ whenever Z represents T, it thus follows from Lemma 3.3 that if $S' \to S$ is *fpqc* covering we may replace X, K, P_0 and P_1 by X', K', P'_0 and P'_1.

Taking in particular $S' = X$, we reduce to the case where X has a section over S. By Lemma 3.2 with $X' = S$, pulling back along such a section does not change T, so that we reduce further to the case where $X = S$ and hence K is an affine group scheme G over S. By base change along an appropriate *fpqc* covering morphism $S' \to S$, we then reduce finally to the case where $P_0 = P_1 = G$. In that case T is represented by G. □

5. Principal bundles

This section contains foundational material on principal bundles, and in particular their connection with transitive affine groupoids. Again it is convenient to work in the category of schemes over a base.

Let S be a scheme and X be a scheme over S with $X \to S$ *fpqc* covering. Let G be an affine group scheme over S. By a right G-scheme over X we mean a scheme P over X with a structure of right G-scheme on P over S such that the action

$$P \times_S G \to P$$

is a morphism over X. A right G-scheme over X is the same as a right G_X-scheme. It is also the same as a right G-scheme equipped with a morphism of right G-schemes to the constant right G-scheme X.

By a *principal G-bundle over X* we mean a right G-scheme over X which is a G_X-torsor. The set of isomorphism classes of principal G-bundles over X will usually be written

$$H^1(X, G)$$

instead of $H^1(X, G_X)$. Similarly the group of cross-sections of G_X will usually be written $H^0(X, G)$ instead of $H^0(X, G_X)$.

We have push forward of principal bundles defined by push forward of torsors, with a strong uniqueness property implied by that for torsors. Using push forward we define a functor $H^1(X, -)$ on affine group schemes over S, which factors through affine group schemes over S up to conjugacy. To a short exact sequence of affine group schemes over S there is associated functorially an exact cohomology sequence of pointed sets.

Let H be a pregroupoid over X/S. Given a scheme P over X equipped with a structure of H-scheme and a structure of right G-scheme over X, it is equivalent to require that the action of H on P be a morphism of right G-schemes over X, or that the action of G on P be a morphism of H-schemes, or that the two morphism

$$H_{[1]} \times_X P \times_S G \to P$$

defined by the actions given by first factoring through $H_{[1]} \times_X P$ or $P \times_S G$ should coincide. We then say that P is an (H,G)*-scheme*. By a *principal* (H,G)*-bundle* we mean an (H,G)-scheme whose underlying right G-scheme is a principal G-bundle over X. A principal (H,G)-bundle is the same as an (H,G_X)-torsor, with G_X regarded as a constant H-group. The set of isomorphism classes of principal (H,G)-bundles will usually be written

$$H^1_H(X,G)$$

instead of $H^1_H(X,G_X)$. Similarly the group of cross-sections of G_X will usually be written $H^0_H(X,G)$ instead of $H^0_H(X,G_X)$.

Let P be a principal (H,G)-bundle, $u : G \to G'$ be a morphism of affine groups schemes over S, P' be a principal G'-bundle over X, and $q : P \to P'$ be a morphism over X compatible with u. Then by Lemma 4.2 there is a unique structure of H-scheme on P' such that P' is a principal (H,G)-bundle and q is a morphism of H-schemes. We say that the principal (H,G)-bundle P' is the push forward of P along u. Using push forward we define a functor $H^1_H(X,-)$ on affine group schemes over S, which factors through affine group schemes over S up to conjugacy. To a short exact sequence of affine group schemes over S there is associated functorially an exact cohomology sequence of pointed sets.

Let P be a principal G-bundle P over X. Then we define the transitive affine groupoid

$$\underline{\mathrm{Iso}}_G(P)$$

over X/S of G-isomorphisms of P as follows: the points of $\underline{\mathrm{Iso}}_G(P)$ in Z above the point (x_0,x_1) of $X \times_S X$ are the G-isomorphism over Z from the pullback of P along x_1 to its pullback along x_0, and the identities and composition of $\underline{\mathrm{Iso}}_G(P)$ those of isomorphisms of principal G-bundles. The existence of $\underline{\mathrm{Iso}}_G(P)$ follows from the case where $X = S$ of Lemma 4.5, with $X \times_S X$ for X and the pullback of P along the ith projection for P_i. We have

$$\mathrm{Aut}_G(P) = H^0(X, \underline{\mathrm{Iso}}_G(P)^{\mathrm{diag}}).$$

If H is a is a pregroupoid over X/S, then a structure of principal (H,G) bundle on P is the same as a structure

$$H \to \underline{\mathrm{Iso}}_G(P) \tag{5.1}$$

of H-groupoid on $\underline{\mathrm{Iso}}_G(P)$. In particular, there is a canonical structure of principal $(\underline{\mathrm{Iso}}_G(P),G)$-bundle on P, with (5.1) the identity.

Let P' be a principal G'-bundle over X, and $h : G \to G'$ be a morphism of affine group schemes over X and $q : P \to P'$ a morphism over X compatible with h. Then there is a unique morphism

$$\underline{\mathrm{Iso}}_h(q) : \underline{\mathrm{Iso}}_G(P) \to \underline{\mathrm{Iso}}_{G'}(P')$$

of groupoids over X such that, for the corresponding structure on P' of principal $(\underline{\mathrm{Iso}}_G(P),G')$-bundle, q is an $\underline{\mathrm{Iso}}_G(P)$-morphism. If H is a pregroupoid over X/S, and P is a principal (H,G)-bundle and P' is a principal (H,G) bundle, then q is a morphism of H-schemes if and only if $\underline{\mathrm{Iso}}_h(q)$ is a morphism of groupoids over H.

Define a category of principal bundles over X/S as follows: the objects are pairs (G,P) with G an affine group scheme over S and P a principal G-bundle over X, and a morphism from (G,P) to $(G'P')$ is a pair (h,q) with $h : G' \to G'$ a morphism of group schemes over S and $q : P \to P'$ a morphism over X compatible with h.

Then $\underline{\mathrm{Iso}}_{-}(-)$ is a functor from this category to transitive affine groupoids over X/S. It is compatible with base extension and pullback, but in general is neither full, faithful, nor essentially surjective.

To each g in $G(S)$ there is associated an *inner automorpism* of (G, P), which acts by conjugation with g on G and as g^{-1} on P. If (h, q) is an inner automorphism of (G, P) then $\underline{\mathrm{Iso}}_h(q)$ is the identity.

Let H be pregroupoid over X/S and P be a principal G-bundle over X/S. Then we have maps

$$\operatorname{Hom}^{\mathrm{conj}}_{X/S}(H, \underline{\mathrm{Iso}}_G(P)) \to H^1_H(X, G) \to H^1(X, G), \tag{5.2}$$

natural in H and (G, P), where the first map sends the conjugacy class of ρ to the class defined by ρ, and the second is defined by discarding the action of H.

Lemma 5.1. *The first map of* (5.2) *is injective, with image the fibre above* $[P]$ *of the second.*

Proof. Immediate from the definitions. □

Let (G, P) be an affine principal bundle over X/S. Then there is a canonical bijection

$$H^1_{\underline{\mathrm{Iso}}_G(P)}(X, G') \xrightarrow{\sim} H^1_G(S, G'), \tag{5.3}$$

natural in the affine group scheme G' over S and compatible with pullback, which when $G' = G$ sends the class of P to the class of G with the right and left actions of G by translation. Indeed by Lemma 4.3, the pullbacks along $P \to X$ and $P \to S$ define natural bijections from the source and target of (5.3) to $H^1_{G_{[P]}}(P, G')$, which when $G' = G$ send the classes of P and G to the class of G_P.

By assigning to the conjugacy class of f to the image under (5.3) of cohomology class defined by f, we obtain a map

$$\operatorname{Hom}^{\mathrm{conj}}_{X/S}(\underline{\mathrm{Iso}}_G(P), \underline{\mathrm{Iso}}_{G'}(P')) \to H^1_G(S, G'), \tag{5.4}$$

which is natural in the affine principal bundles (G, P) and (G', P') and compatible with pullback. It is injective, by Lemma 5.1.

Lemma 5.2. *A morphism* $f : \underline{\mathrm{Iso}}_G(P) \to \underline{\mathrm{Iso}}_{G'}(P')$ *is of the form* $\underline{\mathrm{Iso}}_h(q)$ *for some* (h, q) *if and only if the image of the conjugacy class of* f *under* (5.4) *composed with the canonical map from* $H^1_G(S, G')$ *to* $H^1(S, G')$ *is the base point of* $H^1(S, G')$.

Proof. By compatibility with pullback, we reduce first after pullback along $P \to X$ to the case where $P = G_X$ is trivial, and then after pullback along $X \to S$ to the case where further $X = S$. For any f, the class in $H^1(S, G')$ associated to f is then $[P']$, while $f = \underline{\mathrm{Iso}}_h(q)$ for some (h, q) if and only if P' has a cross-section. □

The map (5.4) has the following description, which renders Lemma 5.2 obvious, and can be seen by reducing to the case where $X = S$ and $P = G$ is trivial: given a morphism f from $\underline{\mathrm{Iso}}_G(P)$ to $\underline{\mathrm{Iso}}_{G'}(P')$, the functor on schemes over S that sends S' to the set of (h', q') with $f_{S'} = \underline{\mathrm{Iso}}_{h'}(q')$, equipped with the action by composition with inner automorphisms of $G(S')$ on the left and $G'(S')$ on the right, is represented by a principal (G, G')-bundle with class in $H^1_G(S, G')$ the image under (5.4) of the conjugacy class of f.

Let H be a pregroupoid over X/S and and x be a section of X over S. We write

$$\widetilde{H}^1_H(X, x, G)$$

for the pointed subset of $H^1_H(X,G)$ consisting of the classes of those P for which P has a section above x. We have a short exact sequence of pointed sets

$$1 \to \widetilde{H}^1_H(X,x,G) \to H^1_H(X,G) \to H^1(S,G) \to 1$$

which is natural in H and G, where the second arrow, defined by discarding the action of H and pulling back along x, is surjective because above the class of the principal G-bundle P over S lies the class of constant (H,G)-bundle P_X.

Given a section x of X over S, we may define as follows a category of affine principal bundles over X/S pointed above x. The objects are triples (G,P,p) with (G,P) an affine principal bundle over X/S and p a section of P over S above x. A morphism from (G,P,p) to (G',P',p') is a morphism (h,q) from (G,P) to (G',P') for which $q(p)=p'$. We have a functor

$$(G,P,p) \mapsto \underline{\mathrm{Iso}}_G(P) \tag{5.5}$$

from this category to the category of transitive affine groupoids over X/S. It is an equivalence, with quasi-inverse the functor

$$K \mapsto (K_{x,x}, K_{-,x}, 1_x)$$

with $K_{x,x}$ the fibre of K above the point (x,x) of the diagonal, and $K_{-,x}$ the fibre of d_1 above x, regarded as a principal $K_{x,x}$-bundle with structural morphism d_0 and action of $K_{x,x}$ defined by composition. Indeed the composition of K defines a structure of principal $(K,K_{x,x})$-bundle on $K_{-,x}$, which gives a natural isomorphism

$$K \xrightarrow{\sim} \underline{\mathrm{Iso}}_{K_{x,x}}(K_{-,x}), \tag{5.6}$$

as can be checked by taking fibres above (x,x). At the same time we have a natural isomorphism

$$(\underline{\mathrm{Iso}}_G(P)_{x,x}, \underline{\mathrm{Iso}}_G(P)_{-,x}, 1_x) \xrightarrow{\sim} (G,P,p)$$

given by evaluating at the base point 1_x.

Lemma 5.3. *Let K be a transitive affine groupoid over X/S and x be a section of X over S.*

(i) *The functor $\widetilde{H}^1_K(X,x,-)$ on affine group schemes over S up to conjugacy is representable.*
(ii) *The functor of* (i) *is represented by G with universal element $[P]$ if and only if the action morphism $K \to \underline{\mathrm{Iso}}_G(P)$ is an isomorphism.*

Proof. By (5.6), a G and principal (K,G)-bundle P trivial above x exist with $K \to \underline{\mathrm{Iso}}_G(P)$ an isomorphism. It is enough to show that G and $[P]$ represent $\widetilde{H}^1_K(X,x,-)$. By Lemma 4.3, we reduce after pullback along x to the case where $X=S$. It then suffices to apply Lemma 5.1 with $X=S$ and $H=K$ to trivial affine principal bundles (G',G'). □

Lemma 5.4. *Let H be a pregroupoid over X/S and $X' \to X$ be a morphism of schemes over S. Suppose either that $X' \to X$ is fpqc covering, or that $X' \to S$ is fpqc covering and H is a transitive groupoid over X/S. Then pullback along $X' \to X$ induces an equivalence from the category of transitive affine groupoids over H to the category of transitive affine groupoids over $H \times_{[X]} [X']$.*

Proof. To prove the full faithfulness, we reduce by base extension along an appropriate $S' \to S$ and faithfully flat descent to the case where X' has a section x' over S. If x' lies above the section x of X, then by (5.6), every transitive affine groupoid over H is isomorphic to $\underline{\mathrm{Iso}}_G(P)$ for some G and principal (H, G)-bundle P with a section p above x. Given also G' and a principal (H, G')-bundle P with a section p' above s, any morphism from $\underline{\mathrm{Iso}}_G(P)$ to $\underline{\mathrm{Iso}}_{G'}(P')$ over H is of the form $\underline{\mathrm{Iso}}_h(q)$ for a unique pair (h, q) with $q(p) = p'$ and q a morphism of H-schemes, and similarly for the pullbacks of P and P' along $X' \to X$. The full faithfulness thus follows from Lemma 3.2.

By the full faithfulness, if K' is a transitive affine groupoid over $H \times_{[X]} [X']$, then a pair consisting of a transitive affine groupoid K over H and an isomorphism from K' to $K \times_{[X]} [X']$ is unique up to unique isomorphism if it exists. To prove the essential surjectivity, we thus reduce by faithfully flat descent after pullback along an appropriate $S' \to S$ to the case where X' has a section over S. Any transitive affine groupoid over X'/S is then by (5.6) a groupoid of isomorphisms, and the essential surjectivity follows from Lemma 4.3. □

Let K be a groupoid over X/S. Call a morphism $K^{\mathrm{diag}} \to J$ of K-groups *compatible with conjugation* if the two actions of K^{diag} on J given by restricting from K to K^{diag} and by restricting the action by conjugation of J along $K^{\mathrm{diag}} \to J$ coincide. We obtain a functor from the category of groupoids over K to the category of those K-morphisms from K^{diag} to a K-group which are compatible with conjugation, by assigning to K' the K-morphism $K^{\mathrm{diag}} \to K'^{\mathrm{diag}}$ defined by restriction, with the action of K on K'^{diag} that defined by restriction of the action of K' by conjugation. It is an equivalence when $X = S$. Any morphism $K^{\mathrm{diag}} \to J$ of K-groups which is *fpqc* covering is compatible with conjugation. Suppose that K is transitive over X/S, and let K' be the pullback of K along a morphism $X' \to X$ with $X' \to S$ an *fpqc* covering morphism. Then by Lemma 3.1, a morphism $K^{\mathrm{diag}} \to J$ of K-groups is compatible with conjugation if and only if the K'-morphism obtained from it by pullback along $X' \to X$ is.

Lemma 5.5. *Let K be a transitive affine groupoid over X/S. Then restriction to the diagonal induces an equivalence from the category of transitive affine groupoids over K to the category of those K-morphisms from K^{diag} to an affine K-group which are compatible with conjugation.*

Proof. When X has a section over S, it is enough to apply Lemmas 3.2 and 5.4 with $X' = S$. The full faithfulness then follows in general by reducing by faithfully flat descent to the case where X has a section after base change along an appropriate $S' \to S$. It follows from the full faithfulness that if a morphism from K^{diag} of affine K-groups extends to a morphism from K of transitive affine groupoids over X, it does so uniquely up to unique isomorphism. The essential surjectivity thus also follows by reducing to the case where X has a section over S. □

Lemma 5.6. *Let (G, P) be an affine principal bundle over X/S. Then $\underline{\mathrm{Iso}}_{-}(-)$ induces an equivalence from the category of affine principal bundles over X/S equipped with a morphism from (G, P) to the category of transitive affine groupoids over $\underline{\mathrm{Iso}}_G(P)$.*

Proof. Pulling back along $P \to X$ and using Lemmas 4.1 and 5.4, we reduce to the case where P is trivial. Pulling (G, G) back along $X \to S$, we then reduce to the case where further $X = S$, which is clear. $\square$

6. Principal bundles under a groupoid

In this section we consider the notion of principal bundle under a transitive affine groupoid. The transitive affine groupoids required for the applications later are those over an algebraically closed extension of a base field. It is again however technically convenient to work here over an arbirary base scheme.

Throughout this section, S is a scheme, X and $\overline{S}$ are schemes over S with structural morphisms *fpqc* covering, and F is a transitive affine groupoid over $\overline{S}/S$. By a *right F-scheme over X* we mean a scheme P over $X \times_S \overline{S}$ together with a structure of right F-scheme on P such that the action

$$P \times_{\overline{S}} F \to P$$

is a morphism over X. A morphism of right F-schemes over X is a morphism of right F-schemes which is a morphism over X. A right F-scheme over X is the same as a right F_X-scheme. It is also the same as a right F-scheme equipped with a morphism of right F-schemes to the constant right F-scheme $X \times_S \overline{S}$ on X. Restriction to F^{diag} defines on any right F-scheme over X an underlying structure of right F^{diag}-scheme over $X \times_S \overline{S}$.

By a *principal F-bundle over X* we mean a right F-scheme over X whose underlying right F^{diag}-scheme over $X \times_S \overline{S}$ is a principal F^{diag}-bundle over $X \times_S \overline{S}$. When $\overline{S} = S$, so that F is an affine group scheme over S, this notion of principal F-bundle over X reduces to that of Section 5.

If $X \to S$ factors through an *fpqc* covering morphism $X \to S'$, we may identify, using the canonical isomorphism

$$P \times_{\overline{S}_{S'}} F_{S'} \xrightarrow{\sim} P \times_{\overline{S}} F,$$

right $F_{S'}$-schemes over X with right F-schemes over X, and principal $F_{S'}$-bundles over X with principal F-bundles over X.

Lemma 6.1. *Let $\overline{S}' \to \overline{S}$ be a morphism with $\overline{S}' \to S$ fpqc covering, and denote by F' the pullback of F along $\overline{S}' \to \overline{S}$. Then pullback along $\overline{S}' \to \overline{S}$ induces an equivalence from the category of principal F-bundles over X to the category of principal F'-bundles over X.*

Proof. By Lemma 3.2, pullback along $X \times_S \overline{S}' \to X \times_S \overline{S}$ induces an equivalence from affine right F_X-schemes to affine right F'_X-schemes. Given an affine right F_X-scheme P, it is thus enough to show that $P \to X \times_S \overline{S}$ is *fpqc*-covering if and only if its pullback is, and that (4.2) with $X \times_S \overline{S}$ for X and $(F^{\mathrm{diag}})_X$ for J is an isomorphism if and only if its pullback is. This is clear from Lemma 3.1, in the case of (4.2) because it is a morphism of G_X-schemes when F_X acts by conjugation on $(F^{\mathrm{diag}})_X$. $\square$

Lemma 6.2. *Let $\overline{X} \to \overline{S}$ be an fpqc covering morphism. Then restriction along the embedding of $\overline{X}$ into $\overline{X} \times_S \overline{S}$ defined by $\overline{X} \to \overline{S}$ induces an equivalence from the category of principal F-bundles over $\overline{X}$ to the category of principal F^{diag}-bundles over $\overline{X}$.*

Proof. Principal F-bundles over $\overline{X}$ are the same as principal $F_{\overline{S}}$-bundles over $\overline{X}$. It thus suffices to take $\overline{S}$, $\overline{S} \times_S \overline{S}$, $\overline{S}$, $\overline{X}$ for S, $\overline{S}$, $\overline{S}'$, X in Lemma 6.1. $\square$

Define as follows a category of transitive affine principal bundles over $X, \overline{S}/S$. The objects are pairs (F, P) with F a transitive affine groupoid over $\overline{S}/S$ and P a principal F-bundle over X. A morphism from (F, P) to (F', P') is a pair (h, q) with h a morphism from F to F' and q a morphism from P to P' compatible with the actions of F and F'.

Lemma 6.3. *Let $\overline{S}' \to \overline{S}$ be a morphism with $\overline{S}' \to S$ fpqc covering. Then the functor induced by pullback along $\overline{S}' \to \overline{S}$ from the category of transitive affine principal bundles over $(X, \overline{S})/S$ to the category of transitive affine principal bundles over $(X, \overline{S}')/S$ is fully faithful.*

Proof. A morphism from (F, P) to (F', P') consists of a morphism of groupoids over $\overline{S}/S$ from F to F' and a morphism of right F-schemes over X from P to P' with the action of F' restricted along $F \to F'$. It thus suffices to apply Lemma 3.2. $\square$

Lemma 6.4. *The forgetful functor from the category of transitive affine principal bundles over $(X, \overline{S})/S$ equipped with a morphism from (G, P) to the category of transitive affine groupoids over $\overline{S}/S$ equipped with a morphism from G is a surjective equivalence.*

Proof. By base change along $\overline{S} \to S$ and faithfully flat descent, we reduce first the full faithfulness and then the surjectivity on objects to the case where $\overline{S}$ has a cross-section over S. Applying Lemmas 6.1 and 6.3 with $\overline{S}' = S$, we reduce further to the case where $\overline{S} = S$, which has been seen in Section 5. $\square$

Given a principal F-bundle P over X and a morphism $h : F \to F'$ of transitive affine groupoids over $\overline{S}/S$, there exists by Lemma 6.4, uniquely up to unique F'-isomorphism, a pair consisting of a principal F'-bundle P' and a morphism $P \to P'$ compatible with h. We say that P' is the *push forward of P along h.*

Let H be a pregroupoid over X/S. Given a principal F-bundle P over X equipped with a structure of H-scheme, it is equivalent to require that the action of H on P be a morphism of right F-schemes over X, or that the action of F on P be a morphism of H-schemes, or that the two morphism

$$H_{[1]} \times_X P \times_{\overline{S}} F \to P$$

defined by the actions given by first factoring through $H_{[1]} \times_X P$ or $P \times_{\overline{S}} F$ should coincide. We then say that P is a *principal (H, F)-bundle.*

Lemma 6.5. *Let P be a principal (H, F)-bundle, $h : F \to F'$ be a morphism of of transitive affine groupoids over $\overline{S}/S$, P' be a principal F'-bundle over X/S, and $q : P \to P'$ be a morphism over X compatible with h. Then there is a unique structure of H-scheme on P' such that P' is a principal (H, F')-bundle and q is a morphism of H-schemes.*

Proof. Apply Lemma 6.4 to the pullbacks of P and P' onto the $H_{[i]}$. $\square$

If P is a principal (H, F)-bundle over X and $h : F \to F'$ is a morphism of transitive affine groupoids over $\overline{S}/S$, then by Lemmas 6.4 and 6.5 there exists, uniquely up to (H, F')-isomorphism, a pair consisting of a principal (H, F')-bundle

P' over X and a morphism $P \to P'$ of H -schemes which is compatible with h. We say that P' is the *push forward of P along $h : F \to F'$*.

Let P be a principal G-bundle P over X. We define the transitive affine groupoid

$$\underline{\mathrm{Iso}}_F(P)$$

over X/S of G-isomorphisms of P as follows: the points of $\underline{\mathrm{Iso}}_F(P)$ in Z above the point (x_0, x_1) of $X \times_S X$ are the F-isomorphism over Z from the pullback of P along x_1 to its pullback along x_0, and the identities and composition of $\underline{\mathrm{Iso}}_G(P)$ those of isomorphisms of principal F-bundles. The existence of $\underline{\mathrm{Iso}}_F(P)$ follows from Lemma 4.5 with $X \times_S X$ for S and $X \times_S X \times_S \overline{S}$ for X and the pullback of P along the the product of ith projection over S with $\overline{S}$ for P_i. A structure of principal (H, F) bundle on P is the same as a structure

$$H \to \underline{\mathrm{Iso}}_F(P) \tag{6.1}$$

of H-groupoid on $\underline{\mathrm{Iso}}_F(P)$. In particular, there is a canonical structure of principal $(\underline{\mathrm{Iso}}_F(P), G)$-bundle on P, with (6.1) the identity.

Let $(h, q) : (F, P) \to (F', P')$ be a morphism of transitive affine principal bundles over $(X, \overline{S})/S$. Then by Lemma 6.5 there is a unique morphism

$$\underline{\mathrm{Iso}}_h(q) : \underline{\mathrm{Iso}}_F(P) \to \underline{\mathrm{Iso}}_{F'}(P')$$

of groupoids over X such that, for the corresponding structure on P' of principal $(\underline{\mathrm{Iso}}_F(P), G')$-bundle, q is an $\underline{\mathrm{Iso}}_F(P)$-morphism. If P is a principal (H, F)-bundle and P' is a principal (H, F') bundle, then q is a morphism of H-schemes if and only if $\underline{\mathrm{Iso}}_h(q)$ is a morphism of groupoids over H.

The assignments $(F, P) \mapsto \underline{\mathrm{Iso}}_F(P)$ and $(h, q) \mapsto \underline{\mathrm{Iso}}_h(q)$ define a functor $\underline{\mathrm{Iso}}_-(-)$ from transitive affine principal bundles over $(X, \overline{S})/S$ to transitive affine groupoids over X/S.

Lemma 6.6. *Let $\overline{S}' \to \overline{S}$ be a morphism with $\overline{S}' \to S$ fpqc covering, and denote by F' the pullback of F along $\overline{S}' \to \overline{S}$. Then pullback along $\overline{S}' \to \overline{S}$ induces an equivalence from the category of principal (H, F)-bundles to the category of principal (H, F')-bundles.*

Proof. Apply Lemma 6.1 to pullbacks along the $H_{[i]} \to X$. □

Lemma 6.7. *Let $X' \to X$ be a morphism, and denote by H' the pullback of H along $X' \to X$. Suppose either that $X' \to X$ is fpqc covering, or that $X' \to S$ is fppqc covering and H is transitive over X/S. Then pullback along $X' \to X$ induces an equivalence from the category of principal (H, F)-bundles to the category of principal (H', F)-bundles.*

Proof. By base change along $\overline{S} \to S$ and faithfully flat descent, we reduce first the full faithfulness and then the essential surjectivity to the case where $\overline{S}$ has a section over S. Applying Lemma 6.6 to pullback along such a serction, we reduce to the case where $\overline{S} = S$. Then F is an affine group scheme over X, and it suffices to apply Lemma 4.3 with $J = F_X$. □

Let K be a transitive affine groupoid over X/S. Then by means of canonical isomorphisms between $X \times_S \overline{S}$, F and K and $\overline{S} \times_S X$, F^{op} and K^{op}, a (K, F)-bundle may be regarded as a (F, K)-bundle, and in particular has an underlying structure of right K-scheme over $\overline{S}$. By a *biprincipal (K, F)-bundle* we mean a principal (K, F)-bundle which is also principal as a K-bundle over $\overline{S}$. A (K, F)-bundle is

biprincipal if and only if is biprincipal as a (F, K)-bundle. When $X = \overline{S}$, we have a canonical biprincipal (F, F)-bundle F, with the left and right actions of F defined by composition.

Lemma 6.8. *Let K be a transitive affine groupoid over X/S and P be a principal (K, F)-bundle. Then P is a biprincipal (K, F)-bundle if and only the canonical morphism $K \to \underline{\mathrm{Iso}}_F(P)$ is an isomorphism.*

Proof. By base change along $\overline{S} \to S$, we reduce to the case where $\overline{S}$ has a section over S. Pulling back along such a section, we reduce to the case where $\overline{S} = S$. By Lemma 3.1, $K \to \underline{\mathrm{Iso}}_F(P)$ is an isomorphism if and only if its restriction to the diagonal is. Replacing K by K^{diag} and F by F_X, we thus reduce to the case where $X = S = \overline{S}$, so that K and F are affine group schemes over S. Pulling back along an appropriate $S' \to S$, we reduce further to the case where P is the trivial F-bundle F, so that $\underline{\mathrm{Iso}}_F(P) = F$. The result is then clear. □

Lemma 6.9. *Let K be a transitive affine groupoid over X/S. Then K is isomorphic to $\underline{\mathrm{Iso}}_F(P)$ for some transitive affine principal bundle (F, P) over $(X, \overline{S})/S$ if and only if a principal K-bundle over $\overline{S}$ exists.*

Proof. A principal K-bundle P over $\overline{S}$ exists if and only if for some transitive affine F over $\overline{S}/S$ a biprincipal (F, K)-bundle exists (take $F = \underline{\mathrm{Iso}}_K(P)$) if and only if for some F a biprincipal (K, F)-bundle exists. The result thus follows from Lemma 6.8. □

The set of isomorphism classes of principal G-bundles over X will be written

$$H^1(X, G).$$

We have a functor $H^1(X, -)$ on transitive affine groupoids up to conjugacy over $\overline{S}/S$. The set of isomorphism classes of principal (H, F)-bundles will be written

$$H^1_H(X, F).$$

Then we have a functor $H^1_H(X, -)$ on transitive affine groupoids up to conjugacy over $\overline{S}/S$. When $\overline{S} = S$, so that F is an affine group scheme over S, the sets $H^1(X, F)$ and $H^1_H(X, F)$ coincide with those of Section 5. For arbitrary $\overline{S}$ however, these sets do not come equipped with a base point, and may even be empty.

Let H be pregroupoid over X/S and P be a principal G-bundle over X/S. Then we have maps

$$\mathrm{Hom}^{\mathrm{conj}}_{X/S}(H, \underline{\mathrm{Iso}}_F(P)) \to H^1_H(X, F) \to H^1(X, F), \tag{6.2}$$

natural in H and (F, P), where the first map sends the conjugacy class of ρ to the class defined by ρ, and the second is defined by discarding the action of H.

Lemma 6.10. *The first map of* (6.2) *is injective, with image the fibre above* $[P]$ *of the second.*

Proof. Immediate from the definitions. □

Let P be a principal F-bundle over X. Then the action of F on P defines a canonical isomorphism of principal F-bundles over P

$$P \times_{\overline{S}} F \xrightarrow{\sim} P \times_X P, \tag{6.3}$$

from the pullback along $P \to \overline{S}$ of the canonical F-bundle F to the pullback along $P \to X$ of P. Indeed (6.3) corresponds under the equivalence of Lemma 6.2 with $\overline{X} = P$ to the canonical isomorphism

$$P \times_{\overline{S}} F^{\mathrm{diag}} \xrightarrow{\sim} P \times_{X \times_S \overline{S}} P$$

of principal F^{diag}-bundles over P defined by the action of F^{diag} on P.

It follows from Lemma 6.7, (6.3), and Lemma 6.8 with $P = K = F$, that pullback along $P \to X$ and $P \to \overline{S}$ defines natural isomorphisms

$$H^1_{\underline{\mathrm{Iso}}_F(P)}(X, -) \xrightarrow{\sim} H^1_{\underline{\mathrm{Iso}}_F(P \times_{\overline{S}} F)}(P, -) \xleftarrow{\sim} H^1_F(\overline{S}, -) \tag{6.4}$$

of functors on transitive affine groupoids over $\overline{S}/S$ up to conjugacy. Further the class in $H^1_{\underline{\mathrm{Iso}}_F(P)}(X, F)$ of P equipped with the canonical structure of principal $(\underline{\mathrm{Iso}}_F(P), F)$-bundle corresponds to the class in $H^1_F(\overline{S}, F)$ of the canonical principal (F, F)-bundle F.

7. Transitive affine groupoids with base the spectrum of a field

In this section k is a field and X is a non-empty k-scheme.

From this section on we work in the category of schemes over k. In particular pregroupoids and groupoids over X will always be understood to be over X/k, and transitivity for a groupoid over X will always be understood as transitivity over X/k. The main result of this section is Theorem 7.3, which implies in particular that when k is algebraically closed, every transitive affine groupoid over X arises from a principal bundle.

The following lemma is taken, with a slight change in notation and hypotheses, from [O'S10, Lemma 1.1.1], to which we refer for the proof. In [O'S10] it was assumed throughout that k is of characteristic 0, and the lemma was proved under the assumption that either the G_λ (i.e. the U_λ of [O'S10, Lemma 1.1.1]) are unipotent or k is algebraically closed. Only the algebraically closed case is required here, and in this case the proof of [O'S10] applies unmodified in arbitrary characteristic.

Lemma 7.1. *Let Λ be a directed preorder, and let $(E_\lambda)_{\lambda \in \Lambda}$ be an inverse system of sets and $(G_\lambda)_{\lambda \in \Lambda}$ be an inverse system of k-groups of finite type. Let there be given an action of $G_\lambda(k)$ on X_λ for each $\lambda \in \Lambda$ such that the transition maps $E_\lambda \to E_{\lambda'}$ are compatible with the actions of $G_\lambda(k)$ and $G_{\lambda'}(k)$. Suppose that k is algebraically closed. Suppose also that for every $\lambda \in \Lambda$ the set E_λ is non-empty, the action of $E_\lambda(k)$ on it is transitive , and the stabiliser of each of its elements is the group of k-points of a k-subgroup of G_λ. Then $\lim_{\lambda \in \Lambda} E_\lambda$ is non-empty, the action of $\lim_{\lambda \in \Lambda} G_\lambda(k)$ on it is transitive , and the stabiliser of each of its elements is the group of k-points of a k-subgroup of $\lim_{\lambda \in \Lambda} G_\lambda$.*

Lemma 7.2. *Suppose that k is algebraically closed. Then $H^1(k, G) = 1$ for every k-group G.*

Proof. Let P be a principal G-bundle over k. Write G as the limit $\lim_{\lambda \in \Lambda} G_\lambda$ of its k-quotients of finite type, and denote by P_λ the push forward of P along $G \to G_\lambda$. Then the (G_λ, P_λ) form an inverse system of affine principal bundles over k. The canonical morphism from P to $\lim_{\lambda \in \Lambda} P_\lambda$ is an isomorphism, because it may be identified after an extension of scalars trivialising P with the canonical morphism from G to $\lim_{\lambda \in \Lambda} G_\lambda$. Since k ia algebraically closed and G_λ is of finite type, P_λ

is trivial. Thus $P_\lambda(k)$ is non-empty and $G_\lambda(k)$ acts simply transitively on it. By Lemma 7.1 with $E_\lambda = P_\lambda(k)$, it follows that $\lim_{\lambda \in \Lambda} P_\lambda(k)$ is non-empty. Thus $P(k)$ is non-empty, so that P is trivial. $\square$

A transitive groupoid over X is faithfully flat over $X \times_k X$, as follows by reducing after base change and pullback to the case where it is constant. Similarly, if F is a transitive affine groupoid over a non-empty k-scheme $\overline{S}$ and P is a principal F-bundle over X, then P is affine and faithfully flat over $X \times_k \overline{S}$. We note that any filtered inverse system of schemes affine and faithfully flat over some base scheme is faithfully flat over the base: the flatness is clear, while the surjectivity reduces after passing to fibres to the case where the base is the spectrum of a field, when it suffices to note that if $1 = 0$ in a filtered colimit of rings A_λ, then $1 = 0$ in some A_λ. Given a filtered inverse system of transitive affine principal bundles (F_λ, P_λ) over $(X, \overline{S})$, it follows in particular from this and (4.2) that $\lim_\lambda P_\lambda$ is a principal $\lim_\lambda G_\lambda$-bundle over X.

It has been seen at the end of Section 3 that if X is the spectrum of a field and K is a transitive affine groupoid of finite type over X, then the quotient K/K' exists for any transitive affine subgroupoid of K.

For the proof of Theorem 7.3 below, we require in a particular case descent for schemes which need not be relatively affine. Suppose that X is a non-empty scheme over an *algebraically closed* field k. Then any descent datum from X to k on a scheme of finite presentation over X is effective. This can be seen by reducing first to the case where X is affine, and then to the case where X is of finite type over k and hence has a k-point, where it is clear. For Theorem 7.3, we need in particular the following fact, which follows easily: if G is k-group, then pullback along the structural morphism of X defines an equivalence from the category of G-schemes of finite type to the category of $G_{[X]}$-schemes of finite presentation.

Theorem 7.3. *Let K be a transitive affine groupoid over X/k and $\overline{k}$ be an algebraically closed extension of k. Then there exists a principal K-bundle over $\overline{k}$.*

Proof. Since principal K-bundles over $\overline{k}$ are the same as principal $K_{\overline{k}}$-bundles over $\overline{k}$, we may suppose that $k = \overline{k}$ is algebraically closed. By Lemma 6.1 with $X = S = \mathrm{Spec}(k)$ and X for $\overline{S}$, we may suppose further that X is the spectrum of a field.

We first construct as follows a filtered inverse system $(G_\lambda)_{\lambda \in \Lambda}$ of k-groups of finite type and an embedding

$$\eta : K \to \lim_{\lambda \in \Lambda} (G_{\lambda [X]})$$

of transitive affine groupoids over X. Choose a set of representatives Γ for representations of K. Since X is the spectrum of a field, we may assume that $\gamma \in \Gamma$ acts on $\mathcal{O}_X^{n(\gamma)}$ for some $n(\gamma)$, so that the action is given by a morphism

$$\eta_\gamma : K \to GL_{n(\gamma)[X]}.$$

Now take for Λ the set of all finite subsets of Γ, ordered by inclusion, for G_λ the product of the $GL_{n(\gamma)}$ for $\gamma \in \lambda$, and for the component of η at λ the morphism with component η_γ at $\gamma \in \lambda$. Then η is the morphism defined by the η_γ to the product of the $GL_{n(\gamma)[X]}$ over $\gamma \in \Gamma$, and hence is an embedding.

Denote by K_λ the image of the component $K \to G_{\lambda[X]}$ of η at λ. The K_λ form an inverse system of transitive affine groupoids over X, and

$$K = \lim_{\lambda \in \Lambda} K_\Lambda.$$

Since X is the spectrum of a field, the quotients $G_{\lambda[X]}/K_\lambda$ exists and are of finite presentation over X. They form an inverse system of schemes over X with transition morphisms compatible with the actions of the $G_{\lambda[X]}$. Since k is algebraically closed, there exists for each λ an G_λ-scheme Z_λ and an $G_{\lambda[X]}$-isomorphism

$$\xi_\lambda : Z_{\lambda X} \xrightarrow{\sim} G_{\lambda[X]}/K_\lambda.$$

Further the Z_λ form in a unique way an inverse system with transition morphisms compatible with the ξ_λ and the actions of the G_λ.

The action of G_λ on Z_λ is transitive because the action of $G_{\lambda[X]}$ on $G_{\lambda[X]}/K_\lambda$ is transitive. Since k is algebraically closed, $G_\lambda(k)$ thus acts transitively on the non-empty set $Z_\lambda(k)$. Hence by Lemma 7.1, the set $\lim_{\lambda\in\Lambda} Z_\lambda(k)$ is non-empty. Let $(z_\lambda)_{\lambda\in\Lambda}$ be an element of this set. Then with G_λ acting on itself by composition, there is a unique morphism of G_λ-schemes

$$\zeta_\lambda : G_\lambda \to Z_\lambda$$

with $\zeta_\lambda(1) = z_\lambda$. The ζ_λ are compatible with the transition morphisms and the actions of the G_λ.

Composing the pullback of ζ_λ onto X with ξ_λ gives a morphism

$$\mu_\lambda = \xi_\lambda \circ \zeta_{\lambda X} = G_{\lambda X} \to G_{\lambda[X]}/K_\lambda$$

of $G_{\lambda[X]}$-schemes. The μ_λ are compatible with the transition morphisms and the actions of the $G_{\lambda[X]}$. Denote by P_λ the inverse image in $G_{\lambda X}$ of the base cross-section (regarded as a closed subscheme) of $G_{\lambda[X]}/K_\lambda$. Since K_λ is the stabiliser of the base cross-section, P_λ is a K_λ-subscheme of $G_{\lambda X}$ which as a right K_λ-scheme over k is principal. The (K_λ, P_λ) then form a filtered inverse system of transitive affine principal bundles over $(k, X)/k$. The limit of the P_λ is thus a principal K-bundle over k. □

Let $\overline{k}$ be an algebraically closed extension of k. Then

$$H^1(\overline{k}, G) = H^1(\overline{k}, G^{\mathrm{diag}}) = 1$$

by Lemma 6.2 with $\overline{X} = \overline{S} = \mathrm{Spec}(\overline{k})$ and Lemma 7.2 with $k = \overline{k}$. Taking $\mathrm{Spec}(k)$, $\mathrm{Spec}(\overline{k})$, G, G', G' for S, X, H, G, P in (6.2), and using Lemma 6.10 thus gives a bijection

$$\mathrm{Hom}^{\mathrm{conj}}_{\overline{k}/k}(G, G') \xrightarrow{\sim} H^1_G(\overline{k}, G') \tag{7.1}$$

which is natural in the transitive affine groupoids G and G' over $\overline{k}$ and which when $G' = G$ sends the identity of G to the class of the principal (G, G)-bundle G.

Lemma 7.4. *The functor $\underline{\mathrm{Iso}}_{-}(-)$ from the category transitive affine principal bundles over $(X, \overline{k})/k$ to the category of transitive affine groupoids over X/k is full and essentially surjective.*

Proof. The essential surjectivity follows from Lemma 6.9 and Theorem 7.3.

To prove that $\underline{\mathrm{Iso}}_{-}(-)$ is full, it is required to show that if (G, P) is a transitive affine principal bundle over $(X, \overline{k})/k$ and P' is a principal $(\underline{\mathrm{Iso}}_G(P), G')$-bundle,

then P' is the push forward of P along some morphism from G to G'. We reduce by (6.4) to the case where $X = \mathrm{Spec}(\overline{k})$ and $K = G$, with P the principal (G, G)-bundle G. This case follows from (7.1). □

Lemma 7.5. *Let K be a transitive affine groupoid over X/k and $\overline{k}$ be an algebraically closed extension of k.*

(i) *The functor $H^1_K(X, -)$ on transitive affine groupoids over $\overline{k}/k$ up to conjugacy is representable.*
(ii) *The functor of* (i) *is represented by G with universal element $[P]$ if and only if the action morphism $K \to \underline{\mathrm{Iso}}_G(P)$ is an isomorphism.*

Proof. By the essential surjectivity of Lemma 7.4, a G and P exist with $K \to \underline{\mathrm{Iso}}_G(P)$ an isomorphism. It is enough to show that G and $[P]$ represent $H^1_K(X, -)$. We reduce using the isomorphism $K \xrightarrow{\sim} \underline{\mathrm{Iso}}_G(P)$ and (6.4) to the case where $X = \mathrm{Spec}(\overline{k})$ and $K = G$, with P the principal (G, G)-bundle G. This case follows from (7.1). □

8. Algebras and modules

In this section k is a field, X is a k-scheme and H is a pregroupoid over X.

In this section we describe the analogue for H-algebras and H-schemes of the usual antiequivalence between commutative quasi-coherent $\mathcal{O}_X$ algebras and schemes affine over X, and of the similar equivalence for modules.

Write ASch_Z for the category of affine schemes over a k-scheme Z and CAlg_Z for the category of commutatatve quasi-coherent $\mathcal{O}_Z$-algebras. For each Z we have an equivalence from $(\mathrm{ASch}_Z)^{\mathrm{op}}$ to CAlg_Z which sends Y to the push forward of $\mathcal{O}_Y$ along the structural morphism of Y. By assigning to Z the categories CAlg_Z and ASch_Z we obtain fibred categories CAlg and ASch over the category of schemes, with pullback functors $Z' \times_Z -$ and $\mathcal{O}_{Z'} \otimes_{\mathcal{O}_Z} -$ along $Z' \to Z$ and pullback isomorphisms the canonical ones. The above equivalences define a morphism

$$\mathrm{ASch}^{\mathrm{op}} \to \mathrm{CAlg}$$

of fibred categories, with the compatibility isomorphisms defined as follows: if Y' is the pullback of Y along $f : Z' \to Z$ and a and a' are the structural morphisms of Y and Y', then the compatibility isomorphism $\mathcal{O}_{Z'} \otimes_{\mathcal{O}_Y} a_*\mathcal{O}_Y \xrightarrow{\sim} a'_*\mathcal{O}_{Y'}$ is adjoint to $a_*\mathcal{O}_Y \to (f \circ a')_*\mathcal{O}_{Y'}$ defined by the morphism $Y' \to Y$ over Z. Choosing quasi-inverses for the above equivalences, we thus obtain a morphism

$$\mathrm{Spec} : \mathrm{CAlg} \to \mathrm{ASch}^{\mathrm{op}}$$

of fibred categories over the category of schemes.

Let $\mathcal{R}$ be a commutative quasi-coherent $\mathcal{O}_Z$-algebra. Then there is a canonical natural isomorphism of $\mathcal{O}_Z$-algebras from $\mathcal{R}$ to the push forward of $\mathcal{O}_{\mathrm{Spec}(\mathcal{R})}$ along the structural morphism of $\mathrm{Spec}(\mathcal{R})$. It defines a morphism of ringed spaces

$$(\mathrm{Spec}(\mathcal{R}), \mathcal{O}_{\mathrm{Spec}(\mathcal{R})}) \to (Z, \mathcal{R}) \tag{8.1}$$

which is natural in $\mathcal{R}$. If $\mathcal{M}$ is an $\mathcal{R}$-module, we denote by $\widetilde{\mathcal{M}}$ its pullback along this morphism of ringed spaces. We write

$$\mathrm{MOD}_{\mathcal{R}}(Z)$$

for the category of quasi-coherent $\mathcal{R}$-modules. Then we have an equivalence

$$\mathrm{MOD}_{\mathcal{R}}(Z) \to \mathrm{MOD}(\mathrm{Spec}(\mathcal{R})) \tag{8.2}$$

defined by $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$.

Define categories CAlgM_Z and $\mathrm{ASch}^{\mathrm{op}}\mathrm{M}_Z$ as follows. An object of CAlgM_Z is a pair $(\mathcal{R}, \mathcal{M})$ with $\mathcal{R}$ a commutative quasi-coherent $\mathcal{O}_Z$-algebra and $\mathcal{M}$ a quasi-coherent $\mathcal{R}$-module, and a morphism from $(\mathcal{R}, \mathcal{M})$ to $(\mathcal{R}', \mathcal{M}')$ is a pair consisting of a morphism $\mathcal{R} \to \mathcal{R}'$ of $\mathcal{O}_X$-algebras and a morphism $\mathcal{M} \to \mathcal{M}'$ of $\mathcal{O}_X$-modules compatible with it, in the sense that the square

$$\begin{tikzcd} \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathcal{R}' \otimes_{\mathcal{O}_X} \mathcal{M}' \arrow[r] & \mathcal{M}' \end{tikzcd} \tag{8.3}$$

defined by $\mathcal{R} \to \mathcal{R}'$, $\mathcal{M} \to \mathcal{M}'$, and the actions of $\mathcal{R}$ on $\mathcal{M}$ and $\mathcal{R}'$ on $\mathcal{M}'$, commutes. The identities and composition of CAlgM_Z are defined componentwise. An object of $\mathrm{ASch}^{\mathrm{op}}\mathrm{M}_Z$ is a pair $(Y, \mathcal{N})$ with Y a scheme over Z and $\mathcal{N}$ a quasi-coherent $\mathcal{O}_Y$-module, and a morphism from $(Y, \mathcal{N})$ to $(Y', \mathcal{N}')$ is a pair (i, ι) with i a morphism $Y' \to Y$ over Z and ι a morphism

$$i^*\mathcal{N} \to \mathcal{N}'$$

of $\mathcal{O}_{Y'}$-modules. The identity of $(Y, \mathcal{N})$ is $(1_Y, 1_{\mathcal{N}})$, and $(i', \iota') \circ (i, \iota)$ is (i'', ι'') with $i'' = i \circ i'$ and ι'' the composite, modulo the pullback isomorphism, of ι and ι'.

We have an equivalence

$$\mathrm{CAlgM}_Z \to \mathrm{ASch}^{\mathrm{op}}\mathrm{M}_Z, \tag{8.4}$$

where $(\mathcal{R}, \mathcal{M})$ is sent to $(\mathrm{Spec}(\mathcal{R}), \widetilde{\mathcal{M}})$ and $(\mathcal{R}, \mathcal{M}) \to (\mathcal{R}', \mathcal{M}')$ is sent to (i, ι) with i the morphism of spectra induced by $\mathcal{R} \to \mathcal{R}'$ and $\iota : i^*\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}'}$ the composite

$$i^*\widetilde{\mathcal{M}} \xrightarrow{\sim} (\mathcal{R}' \otimes_{\mathcal{R}} \mathcal{M})^{\sim} \to \widetilde{\mathcal{M}'}$$

in which the first arrow is the canonical isomorphism which follows from the naturality of (8.1) and the second arrow is obtained by applying the equivalence (8.2) with $\mathcal{R}'$ for $\mathcal{R}$ to $\mathcal{R}' \otimes_{\mathcal{R}} \mathcal{M} \to \mathcal{M}'$ corresponding to $\mathcal{M} \to \mathcal{M}'$.

The assignments $Z \mapsto \mathrm{CAlgM}_Z$ and $Z \mapsto \mathrm{ASch}^{\mathrm{op}}\mathrm{M}_Z$ define fibred categories CAlgM and $\mathrm{ASch}^{\mathrm{op}}\mathrm{M}$ over the category of schemes, with the pullback functors and pullback isomorphisms defined using those for the fibred categories of quasi-coherent sheaves and of relative schemes. We now have a commutative square

$$\begin{tikzcd} \mathrm{CAlgM} \arrow[r] \arrow[d] & \mathrm{ASch}^{\mathrm{op}}\mathrm{M} \arrow[d] \\ \mathrm{CAlg} \arrow[r, "\mathrm{Spec}"] & \mathrm{ASch}^{\mathrm{op}} \end{tikzcd} \tag{8.5}$$

of fibred categories over the category of schemes, with the top and vertical arrows given as follows. The top arrow is defined by the equivalences (8.4), with the compatibility isomorphisms for $f : Z' \to Z$ defined using the compatibility of Spec with pullback together with the commutative square of ringed spaces whose bottom and top arrows are (8.1) and (8.1) with $(Z, \mathcal{R})$ replaced by $(Z', f^*\mathcal{R})$. The left and right arrows of (8.5) are defined by the functors that send $(\mathcal{R}, \mathcal{M})$ to $\mathcal{R}$ and $(Y, \mathcal{N})$ to Y, with the compatibility isomorphisms the identity.

The category of H-objects in CAlg is the category of commutative quasi-coherent H-algebras, which in turn is the category of commutative algebras in $\mathrm{MOD}_H(X)$. Also the category of H-objects in $\mathrm{ASch}^{\mathrm{op}}$ is the same as the dual of the category of affine H^{op}-schemes. Thus Spec defines an equivalence from the category of commutative quasi-coherent H-algebras to the dual of the category of affine H^{op}-schemes.

Let $\mathcal{R}$ be a commutative quasi-coherent H-algebra. Then H-objects in CAlgM with image $\mathcal{R}$ under the left arrow of (8.5) may be identified with a $\mathcal{R}$-modules in $\mathrm{MOD}_H(X)$, as is seen by taking ${d_1}^*\mathcal{R}$ for $\mathcal{R}$ and ${d_0}^*\mathcal{R}$ for $\mathcal{R}'$ in (8.3). A morphism of H-objects in CAlgM lying above the identity of $\mathcal{R}$ is then the same as a morphism of $\mathcal{R}$-modules in $\mathrm{MOD}_H(X)$. We call an $\mathcal{R}$-module in $\mathrm{MOD}_H(X)$ an $(H,\mathcal{R})$-module, and write

$$\mathrm{MOD}_{H,\mathcal{R}}(X)$$

for the category of $(H,\mathcal{R})$-modules.

Let X' be an affine H^{op}-scheme, and write

$$H' = (H^{\mathrm{op}} \times_X X')^{\mathrm{op}}.$$

Then H-objects in $\mathrm{ASch}^{\mathrm{op}}\mathrm{M}$ with image X' under the right arrow of (8.5) may be identified as follows with affine H'-schemes: if

$$i : {d_0}^* X' \xrightarrow{\sim} {d_1}^* X'$$

is the action of H^{op} on X' and $p_i : d_i^* X' \to X'$ is the projection, then the structure of H-object on $(X',\mathcal{M}')$ in $\mathrm{ASch}^{\mathrm{op}}\mathrm{M}$ given by (i,ι) corresponds to the structure

$$(p_1 \circ i)^*\mathcal{M}' \xrightarrow{\sim} i^*{p_1}^*\mathcal{M}' \xrightarrow{\iota} {p_0}^*\mathcal{M}'$$

of H'-module on $\mathcal{M}'$. A morphism of H-objects in $\mathrm{ASch}^{\mathrm{op}}\mathrm{M}$ lying above the identity of X' is then the same as a morphism of H'-modules.

We may regard the H^{op}-scheme X' as an H-scheme in the usual way by inverting the action isomorphism. We then have

$$(H^{\mathrm{op}} \times_X X')^{\mathrm{op}} = H \times_X X'.$$

If we take

$$X' = \mathrm{Spec}(\mathcal{R})$$

for a commutative H-algebra $\mathcal{R}$, then since the functors (8.4) are equivalences, the top arrow of (8.5) thus gives an equivalence from the category of H-objects in CAlgM above $\mathcal{R}$ to the category of H-objects in $\mathrm{ASch}^{\mathrm{op}}\mathrm{M}$ above X'. With the identifications above, we thus obtain an equivalence

$$\mathrm{MOD}_{H,\mathcal{R}}(X) \to \mathrm{MOD}_{H\times_X X'}(X'). \tag{8.6}$$

It sends $\mathcal{M}$ to $\widetilde{\mathcal{M}}$ with an appropriate structure of $(H \times_X X')$-module, and in particular $\mathcal{R}$ to $\mathcal{O}_{X'}$, and it commutes with pullback along a morphisms pregroupoids over X. Considering morphisms with source $\mathcal{R}$ and $\mathcal{O}_{X'}$ thus gives an isomorphism

$$H^0_H(X,\mathcal{V}) \xrightarrow{\sim} H^0_{H\times_X X'}(X',\widetilde{\mathcal{V}}) \tag{8.7}$$

which is natural in $\mathcal{V}$ in $\mathrm{MOD}_{H,\mathcal{R}}(X)$, and commutes with pullback along morphisms to of pregroupoids over X.

By a *representation* of $(H,\mathcal{R})$ we mean an $(H,\mathcal{R})$-module $\mathcal{V}$ such that locally on X, the underlying $\mathcal{R}$-module of $\mathcal{V}$ is a direct summand of a free $\mathcal{R}$-module of

finite type. We denote by $\mathrm{Mod}_{H,\mathcal{R}}(X)$ the category of representations of $(H,\mathcal{R})$. By restriction of (8.6), we obtain an equivalence

$$\mathrm{Mod}_{H,\mathcal{R}}(X) \to \mathrm{Mod}_{H\times_X X'}(X'), \tag{8.8}$$

because $\widetilde{\mathcal{V}}$ is a representation of $H \times_X X'$ if and only if $\mathcal{V}$ is a representation of $(H,\mathcal{R})$.

9. Finite étale algebras

In this section k is a field, X is a non-empty k-scheme, and H is a pregroupoid over X.

This section contains some rather technical results concerning finite étale H-algebras and their filtered colimits, which will be required in Sections 10 and 18.

Let $\mathcal{R}$ be a finite locally free $\mathcal{O}_X$-algebra. Then we have a trace morphism

$$\mathrm{tr} : \mathcal{R} \to \mathcal{O}_X$$

of $\mathcal{O}_X$-modules, which is compatible with pullback. If $\mathcal{R}$ is an H-algebra, then tr is a morphism of H-modules.

Suppose that $\mathcal{R}$ is finite étale. Then locally in the étale topology, $\mathcal{R}$ is isomorphic to a finite product of copies of $\mathcal{O}_X$. If $\mathcal{V}$ and $\mathcal{V}'$ are $\mathcal{O}_X$-modules, and

$$m : \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V} \to \mathcal{V}'$$

is an $\mathcal{O}_X$-homomorphism, there is thus a unique $\mathcal{O}_X$-homomorphism

$$m' : \mathcal{V} \to \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}'$$

such that the composite of the $\mathcal{R}$-homomorphism $\mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{V} \to \mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{V}'$ corresponding to m' with $\mathrm{tr}\otimes_{\mathcal{O}_X}\mathcal{V}'$ is m. If $\mathcal{V}$ and $\mathcal{V}'$ are H-modules and $\mathcal{R}$ is an H-algebra, then m' is a morphism of H-modules if m is. If $\mathcal{V} = \mathcal{V}'$ is an $\mathcal{R}$-module and m is the action of $\mathcal{R}$ on $\mathcal{V}$, then m' is an $\mathcal{R}$-homomorphism, and $m \circ m'$ is the identity of $\mathcal{V}$. For every $\mathcal{R}$-module $\mathcal{V}$, the composite

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{V},\mathcal{R}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{V},\mathcal{R}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{V},\mathcal{O}_X)$$

is an isomorphism, where the first arrow sends an $\mathcal{R}$-homomorphism to its underlying $\mathcal{O}_X$-homomorphism and the second is defined by tr. If $\mathcal{R}$ is an H-algebra, then for every $(H,\mathcal{R})$-module $\mathcal{V}$, the composite

$$\mathrm{Hom}_{H,\mathcal{R}}(\mathcal{V},\mathcal{R}) \to \mathrm{Hom}_{H}(\mathcal{V},\mathcal{R}) \to \mathrm{Hom}_{H}(\mathcal{V},\mathcal{O}_X) \tag{9.1}$$

is an isomorphism, where the first arrow sends an $(H,\mathcal{R})$-homomorphism to its underlying H-homomorphism and the second is defined by tr.

Lemma 9.1. *Let $\mathcal{R}$ be a commutative H-algebra.*

(i) *If $\mathcal{R}$ is finite étale and $\mathcal{V}$ is an $(H,\mathcal{R})$-module, then $\mathcal{V}$ is a direct summand of $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}$.*
(ii) *If $\mathcal{R}$ is finite, locally free, and nowhere 0, and X is of characteristic 0, then $\mathcal{O}_X$ is a direct summand of $\mathcal{R}$.*

Proof. (i) If m above is the action of $\mathcal{R}$ on $\mathcal{V}$, then m and m' are morphisms of $(H,\mathcal{R})$-modules with $m \circ m'$ the identity of $\mathcal{V}$.

(ii) The identity $\mathcal{O}_X \to \mathcal{R}$ composed with $\mathrm{tr} : \mathcal{R} \to \mathcal{O}_X$ is the rank of $\mathcal{R}$, and hence is invertible. ☐

Let H be a pregroupoid over X such that $H^0_H(X,\mathcal{O}_X)$ is a local k-algebra with residue field k. Then given an H-module $\mathcal{V}$, we have a canonical pairing

$$\operatorname{Hom}_H(\mathcal{V},\mathcal{O}_X)\otimes_k H^0_H(X,\mathcal{V}) \to H^0_H(X,\mathcal{O}_X) \to k \tag{9.2}$$

where the first arrow is sends $u\otimes v$ to $u(v)$ and the second arrow is the projection onto the residue field. We write

$${}^{\mathrm{rad}}H^0_H(X,\mathcal{V})$$

for the kernel on the right of this pairing. It is an $H^0_H(X,\mathcal{O}_X)$-submodule of $H^0_H(X,\mathcal{V})$. We have a commutative square

$$\begin{tikzcd} \operatorname{Hom}_H(\mathcal{V}',\mathcal{O}_X)\otimes_k H^0_H(X,\mathcal{V}) \arrow[r] \arrow[d] & \operatorname{Hom}_H(\mathcal{V},\mathcal{O}_X)\otimes_k H^0_H(X,\mathcal{V}) \arrow[d] \\ \operatorname{Hom}_H(\mathcal{V}',\mathcal{O}_X)\otimes_k H^0_H(X,\mathcal{V}') \arrow[r] & k \end{tikzcd} \tag{9.3}$$

associated to any morphism $f:\mathcal{V}\to\mathcal{V}'$ of H-modules. Thus ${}^{\mathrm{rad}}H^0_H(X,-)$ is a subfunctor of $H^0_H(X,-)$ on H-modules.

Let $\mathcal{R}$ be an algebra (not necessarily unitary, commutative, or associative) in $\mathrm{MOD}_H(X)$. Then $H^0_H(X,\mathcal{R})$ is an $H^0_H(X,\mathcal{O}_X)$-algebra, and ${}^{\mathrm{rad}}H^0_H(X,\mathcal{R})$ is a two-sided ideal of $H^0_H(X,\mathcal{R})$. Indeed if we write $\langle u,a\rangle$ for the image of $u\otimes a$ under (9.2), then

$$\langle u(-a),a'\rangle = \langle u,a'a\rangle = \langle u(a'-),a\rangle$$

for every a and a' in $H^0_H(X,\mathcal{R})$ and $u:\mathcal{R}\to\mathcal{O}_X$.

Let $\mathcal{V}$ be a representation of H. Then for every integer $r\geq 0$, the open and closed subscheme of X where $\mathcal{V}$ has rank r is an H-subscheme of X. In particular, if the k-algebra $H^0_H(X,\mathcal{O}_X)$ is indecomposable, then $\mathcal{V}$ has constant rank.

Proposition 9.2. *Let $\mathcal{R}$ be a finite étale H-algebra, and denote by X' the H-scheme* $\operatorname{Spec}(\mathcal{R})$. *Suppose that $H^0_H(X,\mathcal{O}_X)$ and $H^0_H(X,\mathcal{R})$ are local k-algebras with residue field k. Then for every $(H,\mathcal{R})$-module $\mathcal{V}$,* (8.7) *induces an isomorphism*

$${}^{\mathrm{rad}}H^0_H(X,\mathcal{V}) \xrightarrow{\sim} {}^{\mathrm{rad}}H^0_{H\times_X X'}(X',\widetilde{\mathcal{V}})$$

on k-vector subspaces.

Proof. Write $H'=H\times_X X'$. Then we have a diagram

$$\begin{tikzcd} \operatorname{Hom}_{H,\mathcal{R}}(\mathcal{V},\mathcal{R})\otimes_k H^0_H(X,\mathcal{V}) \arrow[r] \arrow[d, "\wr"] & H^0_H(X,\mathcal{R}) \arrow[r] \arrow[d, "\wr"] & H^0_H(X,\mathcal{O}_X) \arrow[d] \\ \operatorname{Hom}_{H'}(\widetilde{\mathcal{V}},\mathcal{O}_{X'})\otimes_k H^0_{H'}(X',\widetilde{\mathcal{V}}) \arrow[r] & H^0_{H'}(X',\mathcal{O}_{X'}) \arrow[r] & k \end{tikzcd}$$

where the vertical isomorphisms are defined using the equivalence (8.6), the left horizontal arrows by composition, the top right arrow by the trace, and the two arrows with target k are the projections onto the residue field. The left square of the diagram commutes by functoriality. To see that the right square commutes, it is enough to show that if s is an element of the maximal ideal of $H^0_H(X,\mathcal{R})$ then $\operatorname{tr}(s)$ lies in the maximal ideal of $H^0_H(X,\mathcal{O}_X)$, or equivalently that $\operatorname{tr}(\overline{s})=0$, where $\overline{s}$ is the image of s in $H^0(\overline{X},\overline{\mathcal{R}})$ with $\overline{X}$ the inverse image in X of the closed point of the spectrum of $H^0_H(X,\mathcal{O}_X)$ and $\overline{\mathcal{R}}=\mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{O}_{\overline{X}}$. Now $\overline{s}$ is finite over k, because by the Cayley–Hamilton Theorem s is finite over $H^0_H(X,\mathcal{O}_X)$. Also $\overline{s}$ is contained in the maximal ideal of the image of $H^0_H(X,\mathcal{R})$ in $H^0(\overline{X},\overline{\mathcal{R}})$. Thus $\overline{s}$ is nilpotent,

so that the cross-section $\mathrm{tr}(\overline{s})$ of $\mathcal{O}_{\overline{X}}$ is locally nilpotent. Since $\mathrm{tr}(\overline{s})$ also lies in the image k of $H^0_H(X, \mathcal{O}_X)$ in $H^0(\overline{X}, \mathcal{O}_{\overline{X}})$, it follows that $\mathrm{tr}(\overline{s}) = 0$, as required.

Modulo the isomorphism given by the composite (9.1), the composite of the top two arrows of the diagram coincides with the first arrow of (9.2). The result follows. □

Let $(X_\lambda)_{\lambda\in\Lambda}$ be a filtered inverse system of schemes affine over X. Then the limit X' of $(X_\lambda)_{\lambda\in\Lambda}$ exists and is affine over X. If H is a pregroupoid over X and $(X_\lambda)_{\lambda\in\Lambda}$ is an inverse system of H-schemes, then X' has a unique structure of H-scheme for which the projections are H-morphisms, and X' is then the limit of $(X_\lambda)_{\lambda\in\Lambda}$ in the category of H-schemes.

Finite limits and finite colimits of finite étale $\mathcal{O}_X$-algebras are finite étale, and are preserved by pullback. The forgetful functor from the category of finite étale H-algebras to the category of finite étale $\mathcal{O}_X$-algebras thus creates limits and colimits. The image $\mathcal{R}''$ of any morphism $\mathcal{R} \to \mathcal{R}'$ of finite étale $\mathcal{O}_X$-algebras is finite étale, and $\mathcal{R}'$ is faithfully flat over $\mathcal{R}''$.

Lemma 9.3. *Any colimit $\mathcal{R}$ of finite étale H-algebras is the colimit of a filtered system $(\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ of finite étale H-algebras with $\mathcal{R}$ faithfully flat over $\mathcal{R}_\lambda$ for each $\lambda \in \Lambda$.*

Proof. Any finite colimit of finite étale $\mathcal{O}_X$-algebras is finite étale. Thus $\mathcal{R}$ is the colimit of a filtered system $(\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ of finite étale H-algebras. Denote by $\overline{\mathcal{R}}_\lambda$ the image of $\mathcal{R}_\lambda \to \mathcal{R}$. Then $\overline{\mathcal{R}}_\lambda$ coincides on any affine open subscheme of X with the image of $\mathcal{R}_\lambda \to \mathcal{R}_{\lambda'}$ for λ' large. Thus $\overline{\mathcal{R}}_\lambda$ is finite étale, $\mathcal{R}$ is the colimit of the $\overline{\mathcal{R}}_\lambda$, and $\mathcal{R}$ is faithfully flat over $\overline{\mathcal{R}}_\lambda$. There is then a unique structure of H-algebra on $\overline{\mathcal{R}}_\lambda$ such that $\mathcal{R}_\lambda \to \overline{\mathcal{R}}_\lambda$ and $\overline{\mathcal{R}}_\lambda \to \mathcal{R}$ are H-morphisms. Replacing $\mathcal{R}_\lambda$ by $\overline{\mathcal{R}}_\lambda$ now gives what is required. □

Lemma 9.4. *Let $(\mathcal{V}_\lambda)_{\lambda\in\Lambda}$ be a filtered system of H-modules with colimit $\mathcal{V}$. Suppose that X is quasi-compact and that $\mathcal{V}_\lambda \to \mathcal{V}$ is universally injective for each $\lambda \in \Lambda$. Then the canonical map*

$$\operatorname*{colim}_{\lambda\in\Lambda} H^0_H(X, \mathcal{V}_\lambda) \to H^0_H(X, \mathcal{V})$$

is bijective.

Proof. Using Lemma 3.2, we may after replacing X by the disjoint union X' of a finite set of affine open subschemes covering X, and H, $(\mathcal{V}_\lambda)$ and $\mathcal{V}$ by their pullbacks onto X', suppose that X is affine.

We have an equaliser diagram

$$H^0_H(X, \mathcal{W}) \longrightarrow H^0(X, \mathcal{W}) \rightrightarrows H^0(H_{[1]}, {d_1}^*\mathcal{W}),$$

natural in the H-module $\mathcal{W}$, with one parallel arrow pullback along d_1 and the other pullback, modulo the action ${d_1}^*\mathcal{W} \xrightarrow{\sim} {d_0}^*\mathcal{W}$ of H on $\mathcal{W}$, along d_0. If T denotes one of $H^0(X, -)$ or $H^0(H_{[1]}, {d_1}^* -)$, then the canonical map from $\mathrm{colim}_{\lambda\in\Lambda} T(\mathcal{V}_\lambda)$ to $T(\mathcal{V})$ is bijective in the first case, because X is affine, and injective in the second case, because its component at each $\lambda \in \Lambda$ is injective. The result follows. □

Lemma 9.5. *Let $(\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ be a filtered system of commutative H-algebras, with colimit $\mathcal{R}$. Suppose that X is quasi-compact and that $\mathcal{R}$ is faithfully flat over $\mathcal{R}_\lambda$*

for each $\lambda \in \Lambda$. Then the canonical map

$$\operatorname*{colim}_{\lambda \in \Lambda} \operatorname{Hom}_{H,\mathcal{R}_\lambda}(\mathcal{R}_\lambda \otimes_{\mathcal{O}_X} \mathcal{V}', \mathcal{R}_\lambda \otimes_{\mathcal{O}_X} \mathcal{V}) \to \operatorname{Hom}_{H,\mathcal{R}}(\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}', \mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V})$$

is bijective for every H-module $\mathcal{V}$ and representation $\mathcal{V}'$ of H.

Proof. Replacing $\mathcal{V}$ by $\mathcal{V}'^{\vee} \otimes_{\mathcal{O}_X} \mathcal{V}$, we may suppose that $\mathcal{V}' = \mathcal{O}_X$. It then suffices to apply Lemma 9.4 with $\mathcal{R}_\lambda \otimes_{\mathcal{O}_X} \mathcal{V}$ for $\mathcal{V}$. $\square$

Lemma 9.6. *Let $(\mathcal{R}_\lambda)_{\lambda \in \Lambda}$ be a filtered system of commutative H-algebras with colimit $\mathcal{R}$. Suppose that $\mathcal{R}$ is faithfully flat over $\mathcal{R}_\lambda$ for each $\lambda \in \Lambda$, that X is quasi-compact and quasi-separated, and that $H_{[1]}$ is quasi-compact. Then every representation of $(H, \mathcal{R})$ is isomorphic to $\mathcal{R} \otimes_{\mathcal{R}_\lambda} \mathcal{V}_0$ for some $\lambda \in \Lambda$ and representation $\mathcal{V}_0$ of $(H, \mathcal{R}_\lambda)$.*

Proof. Let X' be the disjoint union of a finite set of affine open subschemes covering X, then $X' \to X$ is quasi-compact, so that $(H \times_{[X]} [X'])_{[1]}$ is quasi-compact. Using Lemma 3.2, we may thus suppose, after replacing X by X', and H, $\mathcal{R}$, $(\mathcal{R}_\lambda)$ and $\mathcal{W}$ by their pullbacks onto X', that X is affine.

Let $\mathcal{V}$ be a representation of $(H, \mathcal{R})$. The underlying $\mathcal{R}$-module of $\mathcal{V}$ is the image of an idempotent endomorphism e of a free $\mathcal{R}$-module $\mathcal{R}^n$. If we regard ${d_0}^*\mathcal{R}^n$ as a ${d_1}^*\mathcal{R}$-module using the action of H on $\mathcal{R}$, the action of H on $\mathcal{V}$ may then be identified with a morphism of ${d_1}^*\mathcal{R}$-modules

$$\alpha : {d_1}^*\mathcal{R}^n \to {d_0}^*\mathcal{R}^n$$

such that ${d_0}^*e \circ \alpha \circ {d_1}^*e = \alpha$. Taking $H = X$ in Lemma 9.5, with X replaced by $H_{[1]}$ in the case of α, shows that there exists a λ such that e arises from an endomorphism e_0 of the $\mathcal{R}_\lambda$-module ${\mathcal{R}_\lambda}^n$, and α from a morphism of ${d_1}^*\mathcal{R}_\lambda$-modules

$$\alpha_0 : {d_1}^*{\mathcal{R}_\lambda}^n \to {d_0}^*{\mathcal{R}_\lambda}^n$$

with ${d_0}^*{\mathcal{R}_\lambda}^n$ regarded as an ${d_1}^*\mathcal{R}_\lambda$-module using the action of H on $\mathcal{R}_\lambda$. Using the faithful flatness of $\mathcal{R}$ over $\mathcal{R}_\lambda$, we obtain from e_0 an $\mathcal{R}_\lambda$-module $\mathcal{V}_0$ equipped with an isomorphism ι from $\mathcal{R} \otimes_{\mathcal{R}_\lambda} \mathcal{V}_0$ to $\mathcal{V}$, and from α_0 a structure of representation of $(H, \mathcal{R}_\lambda)$ on $\mathcal{V}_0$ such that ι is an $(H, \mathcal{R})$-isomorphism. $\square$

Lemma 9.7. *Let $H \to H'$ be a morphism of pregroupoids over X, and X' be an H'-scheme. Suppose that $X' \to X$ is surjective, that k is of characteristic 0, and that one of the following conditions holds.*

(a) *X' is finite étale.*
(b) *X' is a filtered limit of finite étale H'-schemes, X is quasi-compact and quasi-separated, and $H_{[1]}$ and $H'_{[1]}$ are quasi-compact.*

Then every representation of $H \times_X X'$ is a direct summand of a representation of $H' \times_X X'$ if and only if every representation of H is a direct summand of a representation of H'.

Proof. We may write $X' = \operatorname{Spec}(\mathcal{R})$, where $\mathcal{R}$ is an H'-algebra which is nowhere 0 and which is finite étale if (a) holds and is the colimit of a filtered system $(\mathcal{R}_\lambda)_{\lambda \in \Lambda}$ of finite étale H'-algebras if (b) holds. If (b) holds, we may assume by Lemma 9.3 that $\mathcal{R}$ is faithfully flat over each $\mathcal{R}_\lambda$.

By the equivalence (8.6), it will suffice to prove that every representation of $(H, \mathcal{R})$ is a direct summand of a representation of $(H', \mathcal{R})$ if and only if every representation of H is a direct summand of a representation of H'. The "if" clear,

because Lemmas 9.1(i) and 9.6 show that every representation of $(H, \mathcal{R})$ is a direct summand of $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}$ for a representation $\mathcal{V}$ of H.

To prove the "only if", let $\mathcal{V}$ be a representation of H, and suppose that $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}$ is a direct summand of a representation of $(H', \mathcal{R})$. Then by Lemmas 9.1(i) and 9.6, $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}$ is a direct summand of $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{W}$ for some representation $\mathcal{W}$ of H'. Thus if (a) holds, then $\mathcal{V}$ is by Lemma 9.1(ii) a direct summand of the representation $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{W}$ of H'. If (b) holds, then by Lemma 9.5, $\mathcal{R}_\lambda \otimes_{\mathcal{O}_X} \mathcal{V}$ is a direct summand of $\mathcal{R}_\lambda \otimes_{\mathcal{O}_X} \mathcal{W}$ for some $\lambda \in \Lambda$, so that $\mathcal{V}$ is by Lemma 9.1(ii) a direct summand of the representation $\mathcal{R}_\lambda \otimes_{\mathcal{O}_X} \mathcal{W}$ of H'. □

10. Fundamental groupoids

In this section k is a field, X is a non-empty k-scheme, and H is a pregroupoid over X.

In this section we prove the existence of initial objects in certain full subcategories of the category of transitive affine groupoids over H. These occur as quotients of the universal reductive reductive groupoids Section 15. Their existence is however much easier to prove than that of universal reductive groupoids, because they are unique up to unique isomorphism and not merely up to conjugacy

We say that an H-scheme Y is *H-connected* if it has no open and closed H-subscheme other than $\emptyset$ and X. It is equivalent to require that Y have no decomposition $Y_0 \amalg Y_1$ as the disjoint union of open H-subshemes with neither Y_0 nor Y_1 empty. If k' is a purely inseparable extension of k, then Y is H-connected if and only if $Y_{k'}$ is $K_{k'}$-connected. Also Y is H-connected if and only if Y is $(H \times_X Y)$-connected if and only if $H^0_{H \times_X Y}(Y, \mathcal{O}_Y)$ has no idempotent $\neq 0, 1$.

We say that Y is *geometrically H-connected* if for every finite separable extension k' of k the $H_{k'}$-scheme $Y_{k'}$ is $H_{k'}$-connected. It is equivalent to require that $H^0_{H \times_X Y}(Y, \mathcal{O}_Y)$ should contain no finite étale k-subalgebra other than k. Considering graphs shows that it is also equivalent to require that the functor from finite étale k-schemes to finite étale $(H \times_X Y)$-schemes defined by pullback onto Y be fully faithful.

For any integer $r \geq 0$, the open and closed subscheme of X where a given representation of H has rank r is an H-subsheme of X. Thus if X is H-connected, then any representation of H has constant rank. The same holds for locally free H-schemes of finite type.

Suppose that X is H-connected, and let Y be a finite locally free H-scheme over X. Then Y has constant rank r. Thus Y is the disjoint union of a finite number $\leq r$ of connected open and closed H-subschemes. Further $H^0_{H \times_X Y}(Y, \mathcal{O}_Y)$ has a finite étale k-subalgebra k', of degree $\leq r$, which contains every étale subalgebra. If H' and Y' are obtained from H and Y by a finite separable extension of scalars along a splitting field for k', then Y' is the disjoint union of a finite number $\leq r$ of geometrically H'-connected H'-schemes. If Y is H-connected, then k' is a finite separable extension of k. In this case we may regard Y, with its structure of k'-scheme given by the embedding of k', as an $H_{k'}$-scheme over the k'-scheme $X_{k'}$, and Y is then geometrically $H_{k'}$-connected.

Call an H-scheme *H-proétale* if it is a limit of finite étale H-schemes. Any limit of H-proétale H-schemes is H-proétale. By Lemma 9.3, any H-proétale H-scheme X' is the limit of a filtered system $(X_\lambda)_{\lambda \in \Lambda}$ of finite étale H-schemes with each $X' \to X_\lambda$ faithfully flat. For any such system $(X_\lambda)_{\lambda \in \Lambda}$, and any finite étale H-scheme Y,

the canonical map

$$\operatorname*{colim}_{\lambda \in \Lambda} \operatorname{Hom}_H(X_\lambda, Y) \to \operatorname{Hom}_H(\lim_{\lambda \in \Lambda} X_\lambda, Y) \tag{10.1}$$

is bijective provided that either X is quasi-compact or X is H-connected. Indeed if $Y = \operatorname{Spec}(\mathcal{R})$ and $X_\lambda = \operatorname{Spec}(\mathcal{R}_\lambda)$, then any morphism of H-algebras from $\mathcal{R}$ to $\operatorname{colim} \mathcal{R}_\lambda$ factors through some $\mathcal{R}_\lambda$ on any given quasi-compact open subscheme of X, and hence on X itself when X is H-connected because then both $\mathcal{R}$ and the inverse image of $\mathcal{R}_\lambda$ in $\mathcal{R}$ have constant rank.

Let Y be an H-scheme. We may identify $(H \times_X Y)$-schemes with H-schemes equipped with an H-morphism to Y. Suppose that Y is H-proétale, and that X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact. Then by Lemmas 9.5 and 9.6, for any finite étale $(H \times_X Y)$-scheme Y' there exist finite étale H-schemes Y_0 and X' such that $Y' \to Y$ is the pullback along an H-morphism $Y \to Y_0$ of an H-morphism $X' \to Y_0$. In particular, Y' is an H-proétale H-scheme. From this and the bijections (10.1) it follows that $(H \times_X Y)$-proétale $(H \times_X Y)$-schemes may be identified with morphisms of H-proétale H-schemes with target Y.

Let K be a transitive affine groupoid over X, and X' be a finite étale K-scheme over X which is geometrically K-connected. Then $K \times_X X'$ is transitive affine over X'. This can be seen by reducing in the usual way to the case where $X = \operatorname{Spec}(k)$.

Lemma 10.1. *Let K be a proétale groupoid over H and k' be a finite extension of k. Then K is initial in the category of proétale groupoids over H if and only if $K_{k'}$ is initial in the category of proétale groupoids over $H_{k'}$.*

Proof. Consider the functor from proétale groupoids over H to proétale groupoids over $H_{k'}$ defined by extension of scalars. If k' is separable over k it has a right adjoint given by Weil restriction, while if k' is purely inseparable it is an equivalence. It remains to prove that $K_{k'}$ initial implies K initial for k' finite Galois over k. This is clear by Galois descent. □

Let $\mathcal{C}$ be a full subcategory of the category of transitive affine groupoids over H which is closed under the formation of finite products, filtered limits, transitive affine subgroupoids over H, and affine quotient groupoids. Then the following conditions are equivalent:

(1) $\mathcal{C}$ has an initial object;
(2) for any K of finite type in $\mathcal{C}$, the intersection of two transitive affine sugroupoids over H of K is transitive affine;
(3) for any K and K' of finite type in $\mathcal{C}$, the equaliser of two morphisms from K to K' over H is transitive affine.

Indeed (1) clearly implies (2), and considering the graphs of morphisms shows that (2) implies (3). Assume (3). To see that (1) holds, consider the full subcategory $\mathcal{C}_0$ of $\mathcal{C}$ consisting of those K that have no transitive affine subgroupoid over H other than K. Given K of finite type in $\mathcal{C}$, there is at most one morphism from K_0 to K in $\mathcal{C}$ for any K_0 in $\mathcal{C}_0$, and there is a K_0 in $\mathcal{C}_0$ for which such a morphism exists, because K has a subgroupoid in $\mathcal{C}_0$. Since $\mathcal{C}_0$ is essentially small, the filtered limit $\lim_{K_0 \in \mathcal{C}_0} K_0$ exists, and there exists a unique morphism from it to any K of finite type in $\mathcal{C}$.

Proposition 10.2. *If X is geometrically H-connected, then the category of proétale groupoids over H has an initial object.*

Proof. We show that condition (2) above holds with $\mathcal{C}$ the category of proétale groupoids over H. Let K_1 and K_2 be transitive subgroupoids over H of K. The K/K_i are finite étale K-schemes whose base cross-sections s_i are H-morphisms. The H-morphism to the product of the K/K_i defined by the s_i factors as

$$X \xrightarrow{s} Z \to K/K_1 \times_X K/K_2$$

with Z a K-subscheme of $K/K_1 \times_X K/K_2$ which is K-connected. Then Z is geometrically K-connected by the existence of s, and hence a transitive K-scheme. Thus the stabiliser K_s of s is a transitive subgroupoid over H of K contained in $K_1 \cap K_2$. Hence $K_1 \cap K_2$ is transitive. $\square$

Suppose that X is geometrically H-connected. Then we denote by $\pi_{H,\text{ét}}(X)$ the initial object of the category of finite proétale groupoids over H. For any proétale groupoid K over H, the unique morphism $\pi_{H,\text{ét}}(X) \to K$ over H is faithfully flat if and only if K has no proétale subgroupoid over H other than itself.

Proposition 10.3. *Suppose that for every finite separable extension k' of k*

$$H^1_{H_{k'}}(X_{k'}, (\mathbf{G}_m)_{k'}) = k'^*.$$

Then the category of groupoids of multiplicative type over H has an initial object.

Proof. We show that condition (2) above holds with $\mathcal{C}$ the category of groupoids of multiplicative type over H. Since K is commutative, its diagonal is a constant K-group G_X.

We begin by showing that any two morphisms h_1 and h_2 over H from $1_{[X]}$ to K coincide. We may regard h_1 as a trivialisation of K, so that K is the constant groupoid $G_{[X]}$ over H, and h_1 is the constant morphism. Making a finite separable extension of scalars and embedding G into a finite product of multiplicative groups, we may assume that $G = \mathbf{G}_m$. Writing $G_{[X]} = \underline{\mathrm{Iso}}_G(G_X)$, we may regard h_2 as a trivialisation of the constant principal (H, G)-bundle G_X, i.e. as a pair (P, ι) up to isomorphism with P a principal G-bundle over k and ι an (H, G)-isomorphism from P_X to G_X. Since $G = \mathbf{G}_m$, any such pair is isomorphic to $(G, 1_{G_X})$ by Hilbert's Theorem 90 and the condition on X. Thus $h_2 = h_1$.

Now let K' be a transitive affine subgroupoid over H of K. Since K and K' are commutative, the diagonal of K' is a constant K-subgroup G'_X of the diagonal G_X of K. The image of K' in K/G'_X is then a subgroupoid $1_{[X]}$ of K/G'_X over H, and K' is the inverse image of $1_{[X]}$ in K. By what has already been proved, a subgroupoid $1_{]X]}$ over H of a quotient of K is unique when it exists. Thus transitive affine subgroupoids of K may be identified with those k-subgroups G' of G for which K/G'_X has a subgroupoid $1_{[X]}$ over H. Since

$$K/(G_1 \cap G_2)_X = K/G_{1X} \times_{K/(G_1G_2)_X} K/G_{2X},$$

the required result follows. $\square$

An affine k-group will be said to be *of proétale by multiplicative type* if it is and extension of a proétale k-group by a k-group of multiplicative type, and transitive affine groupoid will be said to be of proétale by multiplicative type if its fibres above the diagonal are.

Proposition 10.4. *Suppose that X is geometrically H-connected and that for every finite separable extension k' of k and non-empty geometrically $H_{k'}$-connected finite étale $H_{k'}$-scheme X' over $X_{k'}$ we have*

$$H_{H_{k'} \times_{X_{k'}} X'}(X', (\mathbf{G}_m)_{k'}) = k'^*.$$

Then the category of groupoids of proétale by multiplicative type over H has an initial object.

Proof. We show that condition (3) above holds with $\mathcal{C}$ the category of groupoids of proétale by multiplicative type over H. To show that the equaliser of morphisms h_1 and h_2 from K to K' over H is transitive affine, we may suppose after a finite separable extension of scalars that

$$K_{\text{ét}} = \underline{\text{Iso}}_G(P)$$

for an étale k-group G and principal (H, G)-bundle P. Then P is finite étale over X, and after a further finite separable extension of scalars we may suppose that P has an H-connected component X' which is geometrically H-connected. Pulling back along $[X'] \to [X]$, we may suppose that there is a morphism of groupoids over H from $[X]$ to $K_{\text{ét}}$. Its composites with $h_{1\text{ét}}$ and $h_{2\text{ét}}$ coincide by Proposition 10.2. Replacing K and K' by $K \times_{K_{\text{ét}}} [X]$ and $K' \times_{K'_{\text{ét}}} [X]$, we may suppose finally that K and K' are of multiplicative type. The equaliser of h_1 and h_2 is then transitive affine by Proposition 10.3. □

Lemma 10.5. *A transitive affine groupoid K over X is proétale if and only if there exists a faithful K-proétale K-scheme.*

Proof. The "if" is clear because K^{con} acts trivially on any finite étale K-scheme.

Suppose that K is proétale. To prove that there exists a faithful K-proétale K-scheme we may suppose that K is finite étale, because if we write K as the filtered limit of its finite étale quotients K_λ, the product of faithful K_λ-proétale K_λ-schemes is a faithful K-proétale K-scheme. Using Lemma 3.2, we may suppose further that $X = \text{Spec}(k')$ for a k-algebra k', then after writing k' as the filtered colimit of its finitely generated k-subalgebras that k' is finitely generated, next after choosing a closed point that k' is a finite extension of k, and finally after taking the separable closure of k in k' that k' is finite separable over k. In that case d_0 defines a structure of finite étale scheme over X on K, and K acts faithfully on itself by composition. □

Let Z be a finite étale scheme over X. Then the groupoid $\underline{\text{Iso}}_X(Z)$ over X, with points in T above (x_0, x_1) the set of isomorphisms from $Z \times_{X,x_1} T$ to $Z \times_{X,x_0} T$ over T, exists, and it is finite étale when Z has constant rank over X. An action of a pregroupoid over X on Z is the same as a morphism over X to $\underline{\text{Iso}}_X(Z)$.

Let X' be a finite étale scheme over X of constant rank. Then $\underline{\text{Iso}}_X(X')$ is a finite étale groupoid over X. Let X_1 be a closed subscheme of X' which is étale of constant rank over X. Then X_1 is also an open subscheme of X', so that $X' = X_1 \amalg X_2$ with X_2 finite étale of constant rank over X. The embeddings of X_1 and X_2 into X' then define an open and closed immersion

$$\underline{\text{Iso}}_X(X_1) \times_{[X]} \underline{\text{Iso}}_X(X_2) \to \underline{\text{Iso}}_X(X') \tag{10.2}$$

of transitive affine groupoids over X. If X' has a structure of H-scheme, then X_1 is a closed H-subscheme of X if and only the action

$$H \to \underline{\mathrm{Iso}}_X(X')$$

of H on X' factors through (10.2).

Lemma 10.6. *Let K be a proétale groupoid over H. Denote by ω the canonical functor from the category of K-proétale K-schemes to the category of H-proétale H-schemes.*

(i) *ω is fully faithful if and only if X is geometrically H-connected and K has no proétale subgroupoid over H other than itself.*
(ii) *ω is an equivalence if and only if K is initial in the category of proétale groupoids over H.*

Proof. (i) Suppose that ω is fully faithful. Then considering constant K-schemes shows that X is geometrically H-connected. Let K' be proétale subgroupoid of K over H. Then K/K' is a transitive proétale K-scheme, and its base cross-section is an H-morphism from X to K/K'. Thus there is a K-morphism from X to K/K', so that $K/K' = X$ and $K' = K$.

Conversely, suppose that X is geometrically connected and that K has no proétale subgroupoid $\neq K$ over H. Then $\pi_{H,\text{ét}}(X)$ exists by Proposition 10.2, and K is a quotient of it. Let X' be a finite étale K-scheme, and X_1 be a closed étale H-subscheme of X' with complement X_2. Then the action of H on X' factors through (10.2). Thus the action of K on X' factors through (10.2), because the unique morphism over H from $\pi_{H,\text{ét}}(X)$ does. Hence X_1 is a K-subscheme of X'. By considering graphs, it follows that the restriction of ω to the category of finite étale K-schemes is fully faithful. Hence ω is fully faithful by the bijections (10.1).

(ii) Suppose that K is initial in the category of proétale groupoids over H. Then ω is fully faithful by Proposition 10.2 and (i). If X' is a finite étale H-scheme, then the action of H on X' factors uniquely through an action of K on X'. Together with the full faithfulness, this shows that ω is essentially surjective.

Conversely suppose that ω is an equivalence. Then by Proposition 10.2 and (i), $\pi_{H,\text{ét}}(X)$ exists and the unique morphism $\pi_{H,\text{ét}}(X) \to K$ over H is faithfully flat. By Lemma 10.5, there exists a faithful $\pi_{H,\text{ét}}(X)$-scheme Z. Since ω is essentially surjective, Z is H-isomorphic to a K-scheme Z'. Then Z and Z' are $\pi_{H,\text{ét}}(X)$-isomorphic by (i) applied to $\pi_{H,\text{ét}}(X)$, so that $\pi_{H,\text{ét}}(X) \to K$ has trivial kernel. □

Lemma 10.7. *Let K be a proétale groupoid over X and X' be a transitive K-scheme. Suppose that either X' is finite or that X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact. Then $K \times_X X'$ is initial in the category of proétale groupoids over $H \times_X X'$ if and only if K is initial in the category of proétale groupoids over H.*

Proof. We may identify $(H \times_X X')$-proétale $(H \times_X X')$-schemes with objects over X' in the category of H-schemes, and similarly with H replaced by K. It is thus enough to show that the functor ω of Lemma 10.6 is an equivalence if and only if it induces an equivalence on categories of objects over X'. The "only if" is immediate. Since X' is the filtered limit of schemes affine and faithfully flat over X, it is faithfully flat over X. The "if" thus follows from faithfully flat descent. □

We say that X is *geometrically simply H-connected* if it is non-empty and geometrically H-connected, and if every finite étale H-scheme is constant.

Suppose that X is geometrically simply H-connected. Then pullback onto X defines an equivalence from the category of finite étale k-schemes to the category of finite étale H-schemes, and hence by the bijections (10.1), from the category of proétale k-schemes to the category of proétale H-schemes.

Let k' be an extension of k. If k' is finite separable, then $X_{k'}$ is geometrically simply $H_{k'}$-connected if and only if X is geometrically simply H-connected. The same holds if k' is purely inseparable, because then pullback defines for every k-scheme Z an equivalence from finite étale schemes over Z to finite étale schemes over $Z_{k'}$. If X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact, the same holds for k' algebraic, by Lemmas 9.5 and 9.6.

Lemma 10.8. *Let K be a transitive affine groupoid over X. Then X is geometrically simply K-connected if and only if K has connected fibres.*

Proof. We reduce first to the case where X is affine, then to the case where k is algebraically closed, and finally to the case where also $X = \mathrm{Spec}(k)$. The result is then clear. □

We call an H-scheme X' a *geometric universal H-cover of X* if X' is H-proétale and geometrically simply $(H \times_X X')$-connected. If X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact, a proétale H-scheme X' is a geometric universal H-cover of X if and only if every H-morphism $X'' \to X'$ with X'' H-proétale is the pullback along the structural morphism of X' of a proétale k-scheme.

A geometric H-universal cover of X need not exist, and need not be unique up to H-isomorphism when it does exist.

Suppose X is geometrically H-connected, and that X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact. Then a $\pi_{H,\text{ét}}(X)$-scheme X' is a geometric universal H-cover of X if and only if it is a geometric universal $\pi_{H,\text{ét}}(X)$-cover of X if and only if X' is a transitive $\pi_{H,\text{ét}}(X)$-scheme and $\pi_{H,\text{ét}}(X) \times_X X'$ is connected if and only if the right $\pi_{H,\text{ét}}(X)^{\text{diag}}$-scheme associated to X' is a $\pi_{H,\text{ét}}(X)^{\text{diag}}$-torsor. If k is separably closed, it follows that a geometric H-universal cover of X exists, and is unique up to H-isomorphism. Similarly if x is a k-point of X, a geometric H-universal cover of X exists with a k-point above x, and is unique up to unique base-point-preserving H-isomorphism.

11. Groupoids and Galois extended groups

In this section k is perfect field and $\overline{k}$ is an algebraic closure of k.

In this section we describe the equivalence between transitive affine groupoids over an algebraic closure $\overline{k}$ of a perfect field k and Galois extended $\overline{k}$-groups, which are affine $\overline{k}$-groups D equipped with extra structure. When k is of characteristic 0, this extra structure is simply a topological extension E of $\mathrm{Gal}(\overline{k}/k)$ by $D(\overline{k})_{\overline{k}}$. This case will be used in Sections 19 and 20 classify principal bundles under a reductive group over curves of genus 0 or 1.

Lemma 11.1. *Suppose that k is algebraically closed.*

(i) *Any surjective k-homomorphism of affine k-groups induces a surjective homomorphism on groups of k-points.*
(ii) *The group of k-points is Zariski dense in any affine k-group.*

Proof. (i) Let $h : G \to G'$ be a surjective k-homomorphism of affine k-groups. To prove that h induces a surjective homomorphism on k-points, we may suppose that G' is reduced and hence that h is faithfully flat. The fibre of h above a k-point of G' is then a principal bundle under $\operatorname{Ker} h$, and has thus a k-point by Lemma 7.2.

(ii) Any affine k-group G is the limit $\lim_\lambda G_\lambda$ of its k-quotients of finite type. The inverse images in G of the open subsets U_λ of the G_λ form a base for the topology of G. Since each non-empty U_λ contains a k-point of G_λ, the required result follows from (i). □

Lemma 11.2. *Suppose that k is algebraically closed of characteristic* 0*. Then for any affine k-group G of finite type, there exists a finitely generated subgroup of $G(k)$ which is Zariski dense in G.*

Proof. Denote by $\mathcal{G}$ the class of those affine k-groups G of finite type for which $G(k)$ has a finitely generated subgroup which is dense in G. Every extension of k-groups in $\mathcal{G}$ lies in $\mathcal{G}$, and $\mathcal{G}$ contains tori, finite k-groups, and the additive group. Any connected reductive k-group G of finite type is generated by a finite set of tori (i.e. has no proper k-subgroup containing them), and hence lies in $\mathcal{G}$. Indeed if G' is maximal among proper k-subgroups of G which are generated by a finite set of tori, there is a torus in G not contained in G' because the tori are dense in G. Since any affine k-group of finite type is a successive extension of k-groups which are either finite, connected reductive, or the additive group, the result follows. □

The forgetful functor $Z \mapsto |Z|$ from the category of local ringed spaces over k to the category of topological spaces has a fully faithful right adjoint $\Theta \mapsto \Theta_{/k}$, where $\Theta_{/k}$ is the local ringed space over k with underlying topological space Θ and structure sheaf the constant sheaf k. For any continuous map $\Theta' \to \Theta$ and morphism $Z \to \Theta_{/k}$ of local ringed spaces over k, the fibre product $\Theta'_{/k} \times_{\Theta_{/k}} Z$ exists: its underlying topological space is $\Theta' \times_\Theta |Z|$ and its structure sheaf is the inverse image of that of Z along the projection. Formation of $\Theta_{/k}$ is compatible with extension of scalars. If Θ is discrete then $\Theta_{/k}$ is a discrete k-scheme, and if Θ is profinite (i.e. compact totally disconnected) then $\Theta_{/k}$ is a profinite k-scheme. In general, however, $\Theta_{/k}$ need not be a scheme, even if Θ is totally disconnected.

Let M be a topological group. Then $M_{/k}$ has a canonical structure of group object in the category of local ringed spaces over k. If M is discrete, then an action of M on a local ringed space Z over k, i.e. a homomorphism from M to $\operatorname{Aut}_k(Z)$, is the same as an action

$$M_{/k} \times Z \to Z$$

of $M_{/k}$ on Z. In general, we say that an action of M on Z is *continuous* if the action

$$(M^d)_{/k} \times Z \to Z$$

of $(M^d)_{/k}$ on Z, where M^d is M rendered discrete, factors (necessarily uniquely) through the epimorphism

$$(M^d)_{/k} \times Z \to M_{/k} \times Z.$$

A continuous action of M on Z is thus the same as an action of $M_{/k}$ on Z.

The right adjoint $\Theta \mapsto \Theta_{/k}$ to the forgetful functor from local ringed spaces over k to topological spaces has itself a right adjoint. It sends Z to $Z(k)$, equipped the topology, which we call the *Krull topology*, with an open base formed by the sets of k-points in a given open subset U of Z at which given sections $f_1, f_2, \ldots, f_n$ of $\mathcal{O}_Z$

above U take given values $\alpha_1, \alpha_2, \dots, \alpha_n$ in k. If Z' is an open local ringed subspace of Z, then $Z'(k)$ is an open subset of $Z(k)$. If Z is a k-scheme which is locally of finite type, then $Z(k)$ is discrete. If Z is an affine k-scheme, then writing Z as the filtered limit of affine k-schemes of finite type shows that $Z(k)$ is the filtered limit of discrete spaces.

More generally, let k' be an extension of k. Then the functor $\Theta \mapsto \Theta_{/k'}$ from topological spaces to local ringed spaces over k has a right adjoint. It sends Z to $Z(k')$, equipped with the topology with an open base formed by the sets of k'-points in a given open subset U of Z at which given sections $f_1, f_2, \dots, f_n$ of $\mathcal{O}_Z$ above U take given values $\alpha_1, \alpha_2, \dots, \alpha_n$ in k'. If Z has a structure of local ringed space over k', then $Z(k')_{k'}$, equipped with the Krull topology, is a subspace of $Z(k')$. If $Z_{k'}$ exists (e.g. if Z is a scheme), then the topological space $Z(k')$ coincides with $Z_{k'}(k')_{k'}$.

Suppose now that k is perfect, and let $\overline{k}$ be an algebraic closure of k. For any k-scheme X, the group $\mathrm{Gal}(\overline{k}/k)$ acts on the left on the set $X(\overline{k})$ of $\overline{k}$-points of X, with σ in $\mathrm{Gal}(\overline{k}/k)$ sending w in $X(\overline{k})$ to

$$^{\sigma}w = w \circ \mathrm{Spec}(\sigma).$$

This action is continuous for the Krull topology on $X(\overline{k})$, because it sends basic open sets to basic open sets, and every basic open set is fixed by an open subgroup of $\mathrm{Gal}(\overline{k}/k)$. We have

$$f(^{\sigma}w) = {}^{\sigma}(f(w))$$

for any k-morphism $f : X \to Y$.

The category of groupoids over $\overline{k}$ has a final object

$$\Gamma = \mathrm{Spec}(\overline{k}) \times_k \mathrm{Spec}(\overline{k}),$$

which is also final in the category of graphs over $\overline{k}$. We may regard graphs over $\overline{k}$ as $\overline{k}$-schemes using d_0. Then we have an isomorphism of $\overline{k}$-schemes

$$\mathrm{Gal}(\overline{k}/k)_{/\overline{k}} \xrightarrow{\sim} \Gamma \tag{11.1}$$

which induces on $\overline{k}$-points over $\overline{k}$ a homeomorphism

$$\mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} \Gamma(\overline{k})_{\overline{k}} \tag{11.2}$$

sending σ to $(1_{\mathrm{Spec}(\overline{k})}, \mathrm{Spec}(\sigma))$. This can be seen by taking the limit over finite Galois extensions $k_0 \subset \overline{k}$ of k of isomorphisms of finite discrete $\overline{k}$-schemes from $\mathrm{Spec}(\overline{k}) \times_k \mathrm{Spec}(k_0)$ to $\mathrm{Gal}(k_0/k)$.

Let F be a transitive affine groupoid over $\overline{k}$. Then the composite of the continuous map $F(\overline{k})_{\overline{k}} \to \Gamma(\overline{k})_{\overline{k}}$ defined by (d_0, d_1) with the inverse of the homeomorphism (11.2) is a continuous map

$$\gamma : F(\overline{k})_{\overline{k}} \to \mathrm{Gal}(\overline{k}/k)$$

such that $d_1(u) = \mathrm{Spec}(\gamma(u))$ for every u in $F(\overline{k})_{\overline{k}}$. Let u and v be elements of $F(\overline{k})_{\overline{k}} \subset F(\overline{k})$. Define the product $u.v$ of u and v as

$$u.v = u \circ {}^{\gamma(u)}v, \tag{11.3}$$

where $\circ$ denotes the composition of F. With this product and the Krull topology, $F(\overline{k})_{\overline{k}}$ is a topological group. To check for example the continuity of the product,

note that the product composed with the embedding into $F(\overline{k})$ of the subspace $F(\overline{k})_{\overline{k}}$ factors as

$$F(\overline{k})_{\overline{k}} \times F(\overline{k})_{\overline{k}} \to F(\overline{k}) \times_{\mathrm{Spec}(\overline{k})(\overline{k})} F(\overline{k}) \to F(\overline{k}),$$

where the first arrow sends (u, v) to $(u, {}^{\gamma(u)}v)$ and is continuous because γ and the action of $\mathrm{Gal}(\overline{k}/k)$ on $F(\overline{k})$ are continuous, and the second arrow is the composition and is continuous because it is defined by a morphism of k-schemes. The map γ is now a continuous homomorphism.

It will be shown in Lemma 11.3 below that for any transitive affine groupoid G over $\overline{k}$ there exists a section $t : \Gamma \to G$ of the morphism of $\overline{k}$-schemes $G \to \Gamma$. After translating, t can be chosen to be the identity above the diagonal. Any section t factors uniquely through the counit

$$(F(\overline{k})_{\overline{k}})_{/\overline{k}} \to F \tag{11.4}$$

for F in the category of local ringed spaces over $\overline{k}$. Further any point of F in a k-scheme can be written uniquely in the form

$$d \circ t(u) \tag{11.5}$$

for points d of F^{diag} and u of Γ with $d_0(u) = d_1(d) = d_0(d)$.

For the proof of Lemma 11.3 we require the following fact: if Z is a filtered limit of reduced finite schemes over an algebraically closed field and $Y \to Z$ is a morphism which is affine and of finite presentation, then Y is locally constant over Z. In particular, if $Y \to Z$ is surjective, it is a retraction.

Lemma 11.3. *Let F be a transitive affine groupoid over $\overline{k}$. Then the morphism of $\overline{k}$-schemes $(d_0, d_1) : F \to \Gamma$ is a retraction.*

Proof. Consider the partially ordered set $\mathcal{P}$ of pairs (N, s) with N an F-subgroup of F^{diag} and s a section of the morphism of schemes $(d_0, d_1) : F/N \to \Gamma$, where $(N, s) \leq (N', s')$ when $N' \subset N$ and s' lifts s. Since any filtered limit of quotients of F is a quotient of F, the set $\mathcal{P}$ is inductively ordered.

Let (N, s) be a maximal element of $\mathcal{P}$. It is enough to show that $N = 1$. Let N_0 be an F-subgroup of F^{diag} with F/N_0 of finite type, and write N' for $N \cap N_0$. Then the morphism $F/N' \to F/N$ is of finite presentation. Now the $\overline{k}$-scheme Γ is reduced and profinite. Thus by the remark above, $F/N' \to F/N$ has a section above the image of s, which defines a lifting s' of s. Then $(N', s') \geq (N, s)$, so that $N' = N$. Thus $N \subset N_0$ for any N_0 with F/N_0 of finite type, so that $N = 1$ as required. $\square$

Let E' and E'' be topological groups. By an *extension*

$$1 \mapsto E'' \to E \to E' \to 1$$

of E' by E'' we mean a topological group E together with continuous homomorphisms $E \to E'$ and $E'' \to E$ such that $E'' \to E$ is a topological isomorphism onto the kernel of $E \to E'$ and $E \to E'$ is locally on E' a retraction of topological spaces. It is equivalent to require that E be a principal E''-bundle over E' in the usual topological sense for the action by right translation of E'' on E. This notion will be used in what follows only when E' is profinite, and in that case $E \to E'$ has necessarily a section globally when is has one locally on E'.

Given $\overline{k}$-schemes Z and Z' and σ in $\mathrm{Gal}(\overline{k}/k)$, define a *$\sigma$-morphism from Z to Z'* as a k-morphism f from Z to Z' such that the square formed by f, the structural

morphisms of Z and Z', and $\mathrm{Spec}(\sigma^{-1})$, commutes. Such a σ-morphism f induces a map from $Z(\overline{k})_{\overline{k}}$ to $Z'(\overline{k})_{\overline{k}}$, which sends z in $Z(\overline{k})_{\overline{k}}$ to the unique z' in $Z'(\overline{k})_{\overline{k}}$ for which the square formed by f, z, z', and $\mathrm{Spec}(\sigma^{-1})$, commutes. Equivalently, if we identify $Z(\overline{k})_{\overline{k}}$ and $Z'(\overline{k})_{\overline{k}}$ with subsets of the topological spaces Z and Z', the map is the one induced on these subsets. The composite of a σ-morphism from Z to Z' with a σ'-morphism from Z' to Z'' is a $(\sigma'\sigma)$-morphism from Z to Z''. If a σ-morphism is an isomorphism of k-schemes, its inverse is a σ^{-1}-morphism, and we speak a σ-isomorphism, or a σ-automorphism when the source and target coincide.

Given $\overline{k}$-groups D and D', a σ-morphism $h : D \to D'$ of $\overline{k}$-schemes will be called a *σ-morphism of $\overline{k}$-groups* if the squares formed by $h \times_{\mathrm{Spec}(\sigma^{-1})} h$, the products of D and D', and h, and by $\mathrm{Spec}(\sigma^{-1})$, the identities of D and D', and h, both commute. For such an h, the map from $D(\overline{k})_{\overline{k}}$ to $D'(\overline{k})_{\overline{k}}$ induced by h is a group homomorphism.

We may identify a σ-morphism from Z to Z' with a morphism of $\overline{k}$-schemes

$$\mathrm{Spec}(\sigma)^* Z \to Z'$$

from the pullback of Z along $\mathrm{Spec}(\sigma)$ to Z'. A σ-morphism $D \to D'$ of $\overline{k}$-groups is then a $\overline{k}$-homomorphism from $\mathrm{Spec}(\sigma)^* D$ to D'.

Denote by $\overline{\mathcal{T}}_k$ the full subcategory of the category of local ringed spaces over k consisting of of those Z for which $\mathcal{O}_Z$ is k-isomorphic to the constant k-sheaf $\overline{k}$. The morphism $f^{-1}\mathcal{O}_Z \to \mathcal{O}_{Z'}$ induced on structure sheaves by any morphism $f : Z' \to Z$ in $\overline{\mathcal{T}}_k$ is an isomorphism, because any k-homomorphism between algebraic closures of k is an isomorphism. It follows that pullbacks along any morphism in $\overline{\mathcal{T}}_k$ exist in the category of local ringed spaces over k, and that $Z' \times_Z Z''$ lies in $\overline{\mathcal{T}}_k$ when Z, Z' and Z'' do. Further binary products, and hence non-empty finite limits, of local ringed spaces over k in $\overline{\mathcal{T}}_k$ exist and lie in $\overline{\mathcal{T}}_k$, because by (11.1) the product of $\mathrm{Spec}(\overline{k})$ with itself lies in $\overline{\mathcal{T}}_k$.

Let E be an extension of the profinite group $\mathrm{Gal}(\overline{k}/k)$ by a topological group E'. By composing the projection from $E_{/\overline{k}}$ to $\mathrm{Gal}(\overline{k}/k)_{/\overline{k}}$ with (11.1), we obtain on $E_{/\overline{k}}$ a structure of graph in $\overline{\mathcal{T}}_k$ over $\overline{k}$, with $E_{/\overline{k}} \to \Gamma$ a retraction. There is then a unique morphism

$$E_{/\overline{k}} \times_{\overline{k}} E_{/\overline{k}} \to E_{/\overline{k}}$$

of graphs in $\overline{\mathcal{T}}_k$ over $\overline{k}$ whose underlying map of topological spaces is the product of E. It defines on $E_{/\overline{k}}$ a structure of groupoid in $\overline{\mathcal{T}}_k$ over $\overline{k}$, with diagonal $E'_{/\overline{k}}$. If Z is a $\overline{k}$-scheme, the projection of $E_{/\overline{k}}$ onto $E_{/k}$ defines an isomorphism

$$E_{/\overline{k}} \times_{\overline{k}} Z \xrightarrow{\sim} E_{/k} \times Z \tag{11.6}$$

of local ringed spaces over $\overline{k}$. Using this isomorphism, we may identify an action of the groupoid $E_{/\overline{k}}$ on Z with a continuous action of the topological group E on the underlying k-scheme of Z such that e in E above σ in $\mathrm{Gal}(\overline{k}/k)$ acts as a σ-automorphism of Z. For a transitive affine groupoid F over $\overline{k}$, the counit (11.4) is compatible with the groupoid structures.

By a *Galois extended $\overline{k}$-group* (D, E) we mean an affine $\overline{k}$-group D, an extension

$$1 \to D(\overline{k})_{\overline{k}} \to E \to \mathrm{Gal}(\overline{k}/k) \to 1$$

of topological groups, and a continuous action of E on the underlying k-scheme of D with e in E above σ in $\mathrm{Gal}(\overline{k}/k)$ acting as a σ-automorphism a_e of the $\overline{k}$-group D, such that

(a) the automorphism of $D(\overline{k})_{\overline{k}}$ induced by a_e is conjugation by e;
(b) for e in $D(\overline{k})_{\overline{k}}$, the $\overline{k}$-automorphism a_e of D is conjugation by e.

A morphism of Galois extended $\overline{k}$-groups from (D, E) to (D', E') is a pair (h, l) with h a $\overline{k}$-homomorphism from D to D' and l a continuous homomorphism from E to E', such that the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & D(\overline{k})_{\overline{k}} & \longrightarrow & E & \longrightarrow & \mathrm{Gal}(\overline{k}/k) & \longrightarrow & 1 \\
 & & \Big\downarrow h & & \Big\downarrow l & & \Big\| & & \\
1 & \longrightarrow & D'(\overline{k})_{\overline{k}} & \longrightarrow & E' & \longrightarrow & \mathrm{Gal}(\overline{k}/k) & \longrightarrow & 1
\end{array}
\tag{11.7}
$$

commutes, and such that h intertwines with the action of e on D and $l(e)$ on D' for any e in E.

If we identify a continuous action of E on the underlying k-scheme of D with an action of $E_{/k}$ on D, and use (11.6), then a continuous action of E on D with e above σ acting as a σ-automorphism of the $\overline{k}$-group D may be identified with an action

$$\alpha : E_{/\overline{k}} \times_{\overline{k}} D \to D \tag{11.8}$$

of the groupoid $E_{/\overline{k}}$ in $\overline{\mathcal{T}}_{\overline{k}}$ over $\overline{k}$ on the $\overline{k}$-group D. If we write

$$\varepsilon : (D(\overline{k})_{\overline{k}})_{/\overline{k}} \to D \tag{11.9}$$

for the counit, condition (a) is then equivalent to the condition

$$\alpha(w, \varepsilon(v)) = \varepsilon(w \circ v \circ w^{-1}) \tag{11.10}$$

on points w of $E_{/\overline{k}}$ and v of its diagonal $(D(\overline{k})_{\overline{k}})_{/\overline{k}}$, and (b) is equivalent to the condition

$$\alpha(v, d) = \varepsilon(v) d \varepsilon(v)^{-1} \tag{11.11}$$

on points v of $(D(\overline{k})_{\overline{k}})_{/\overline{k}}$ and d of D. Given also a Galois extended $\overline{k}$-group (D', E') with action α' and counit ε', a morphism of Galois extended $\overline{k}$-groups from (D, E) to (D', E') may be identified with a $\overline{k}$-homomorphism $h : D \to D'$ together with a morphism $\lambda : E_{/\overline{k}} \to E'_{/\overline{k}}$ of groupoids in $\overline{\mathcal{T}}_k$ over $\overline{k}$, such that

$$h(\varepsilon(v)) = \varepsilon'(\lambda(v)) \tag{11.12}$$

for points v of $(D(\overline{k})_{\overline{k}})_{/\overline{k}}$, and

$$h(\alpha(w, d)) = \alpha'(\lambda(w), h(d)). \tag{11.13}$$

for points w of $E_{/\overline{k}}$ and d of D.

To every transitive affine groupoid F over $\overline{k}$ is associated a Galois extended $\overline{k}$-group $(F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$, where the first arrow in

$$1 \to F^{\mathrm{diag}}(\overline{k})_{\overline{k}} \to F(\overline{k})_{\overline{k}} \to \mathrm{Gal}(\overline{k}/k) \to 1$$

is the embedding, the second is γ, and the action $\mathrm{Spec}(\gamma(v))^* F^{\mathrm{diag}} \xrightarrow{\sim} F^{\mathrm{diag}}$ on F^{diag} of v in $F(\overline{k})_{\overline{k}}$ is the action at v of F on F^{diag}. The first arrow is a topological isomorphism onto the kernel of γ because it is defined by the embedding of the

fibre of F above a $\overline{k}$-point of Γ over $\overline{k}$, and γ is a retraction of topological spaces by Lemma 11.3. The action of $F(\overline{k})_{\overline{k}}$ on F^{diag} arises from the action α of the form (11.8) with $D = F^{\mathrm{diag}}$ and $E = F(\overline{k})_{\overline{k}}$, given by restricting the action of F on F^{diag} along the counit (11.4). The action is thus continuous with e above σ acting as a σ-automorphism of the $\overline{k}$-group F^{diag}. It satisfies (a) because (11.10) holds for α by naturality of the counit, and (b) because (11.11) is immediate. We obtain a functor from the category of transitive affine groupoids over $\overline{k}$ to the category of Galois extended $\overline{k}$-groups by assigning to $F \to F'$ the pair (h, l) with h the restriction to the diagonals and l the map induced on $\overline{k}$-points over $\overline{k}$.

Proposition 11.4. *The functor from the category of transitive affine groupoids over $\overline{k}$ to the category of Galois extended $\overline{k}$-groups that sends F to $(F^{\mathrm{diag}}, G(\overline{k})_{\overline{k}})$ is an equivalence of categories.*

Proof. Let F and F' be transitive affine groupoids over $\overline{k}$, and let $h : F^{\mathrm{diag}} \to F'^{\mathrm{diag}}$, and $\lambda : (F(\overline{k})_{\overline{k}})_{/\overline{k}} \to F'(\overline{k})_{\overline{k}})_{/\overline{k}}$ define a morphism from $(F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$ to $(F'^{\mathrm{diag}}, F'(\overline{k})_{\overline{k}})$. To prove the full faithfulness, it is to be shown that there is a unique morphism $f : F \to F'$ of groupoids over $\overline{k}$ which induces h and λ.

By Lemma 11.3, the morphism $F \to \Gamma$ over $\overline{k}$ has a section $t : \Gamma \to F$, which we may suppose is the identity above the diagonal. Factor t over $\overline{k}$ as $t_0 : \Gamma \to (F(\overline{k})_{\overline{k}})_{/\overline{k}}$ followed by the counit (11.4), and write t' for $\lambda \circ t_0$ composed with the counit for F'. By the factorisation (11.5), an f inducing h and λ is unique if it exists, because it must send $d \circ t(u)$ to $h(d) \circ t'(u)$ for points d of G^{diag} and u of Γ with $d_0(u) = d_1(d)$. The morphism f so defined preserves the identity, and it preserves composition because

$$h(t(u) \circ d \circ t(u)^{-1}) = t'(u) \circ h(d) \circ t'(u)^{-1}$$

by (11.13) and

$$h(t(u) \circ t(u') \circ t(u \circ u')^{-1}) = t'(u) \circ t'(u') \circ t'(u \circ u')^{-1}$$

by (11.12). It is thus a morphism of groupoids over $\overline{k}$.

Let (D, E) be a Galois extended $\overline{k}$-group. To prove the essential surjectivity, it is to be shown that there exists a transitive affine groupoid F over $\overline{k}$ with $(F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$ isomorphic to (D, E).

The continuous map $E \to \mathrm{Gal}(\overline{k}/k)$ has a section which sends 1 to 1. Applying $(-)_{/\overline{k}}$ and using (11.1), we obtain a section

$$s : \Gamma \to E_{/\overline{k}}$$

to the morphism $E_{/\overline{k}} \to \Gamma$ of graphs over $\overline{k}$, which is the identity above the diagonal. With ε the counit (11.9), define a morphism $\delta : \Gamma \times_{\overline{k}} \Gamma \to D$ by

$$\delta(u, u') = \varepsilon(s(u) \circ s(u') \circ s(u \circ u')^{-1}),$$

and take for F the graph

$$F = D \times_{\overline{k}} \Gamma$$

over $\overline{k}$, with identity $(1, 1)$ and composition $\circ : F \times_{\overline{k}} F \to F$ defined by

$$(d, u) \circ (d', u') = (d\alpha(s(u), d')\delta(u, u'), u \circ u').$$

The composition is associative because

$$\delta(u, u')\delta(u \circ u', u'') = \alpha(s(u), \delta(u', u''))\delta(u, u' \circ u'')$$

by (11.10), and

$$\delta(u,u')\alpha(s(u\circ u'),d'')\delta(u,u')^{-1} = \alpha(s(u),\alpha(s(u'),d''))$$

by (11.11) and the associativity of the action α. Also inverses exist:

$$(d,u)^{-1} = (1,u)^{-1}\circ(d,1)^{-1} = (\varepsilon(s(u)^{-1}\circ s(u^{-1})^{-1}),u^{-1})\circ(d^{-1},1).$$

Thus F is a groupoid over $\overline{k}$. It is transitive affine, because $F\to\Gamma$ is the projection. By compatibility of (11.4) with composition, the composition in

$$(F(\overline{k})_{\overline{k}})_{/\overline{k}} = (D(\overline{k})_{\overline{k}})_{/\overline{k}}\times_{\overline{k}}\Gamma$$

is given by

$$(v,u)\circ(v',u') = (v\circ s(u)\circ v'\circ s(u')\circ s(u\circ u')^{-1}, u\circ u').$$

We thus have an isomorphism $(F(\overline{k})_{\overline{k}})_{/\overline{k}} \xrightarrow{\sim} E_{/\overline{k}}$ sending (v,u) to $v\circ s(u)$. Along with $F^{\mathrm{diag}}\xrightarrow{\sim} D$ sending $(d,1)$ to d, this gives an isomorphism from $(F^{\mathrm{diag}},F(\overline{k})_{\overline{k}})$ to (D,E). Indeed (11.12) is clear, while (11.13) holds for points $(1,u)$ of $(F(\overline{k})_{\overline{k}})_{/\overline{k}}$ by (11.10) and for points $(v,1)$ by (11.11), and hence for any $(v,u)=(v,1)\circ(1,u)$ by associativity of the actions. □

Define a *loosely Galois extended* $\overline{k}$*-group* as a pair (D,E) with D an affine $\overline{k}$-group and E an extension $\mathrm{Gal}(\overline{k}/k)$ by $D(\overline{k})_{\overline{k}}$, such that for e in E above σ in $\mathrm{Gal}(\overline{k}/k)$, the automorphism $d\mapsto e.d.e^{-1}$ of $D(\overline{k})_{\overline{k}}$ is induced by a σ-automorphism of the $\overline{k}$-group D. A morphism of loosely Galois extended $\overline{k}$-groups from (D,E) to (D',E') is a pair (h,j) with h an $\overline{k}$-homomorphism from D to D' and j a continuous homomorphism from E to E', such that the diagram (11.7) commutes.

If D and D' are affine $\overline{k}$-groups with D reduced, then two σ-morphisms of $\overline{k}$-groups $D\to D'$ which induce the same homomorphism $D(\overline{k})_{\overline{k}}\to D'(\overline{k})_{\overline{k}}$ coincide: their equaliser is a closed subscheme of D containing the subset $D(\overline{k})_{\overline{k}}$, which is dense by Lemma 11.1(ii). It follows that for (D,E) a loosely Galois extended $\overline{k}$ with D reduced, there is a for each e in E above σ unique σ-automorphism a_e of the k-group D which induces conjugation by G on $D(\overline{k})_{\overline{k}}$. Explicitly,

$$a_e({}^{\sigma}d) = e.d.e^{-1}$$

for each d in $D(\overline{k})_{\overline{k}}$. Further the a_e satisfy conditions (a) and (b) above, and hence define a structure of Galois extended $\overline{k}$-group provided that the action $e\mapsto a_e$ is continuous. Similarly if (D,E) and (D',E') are Galois extended $\overline{k}$-groups and D is reduced, then any morphism of loosely Galois extended $\overline{k}$-groups from (D,E) to (D',E') is a morphism of Galois extended $\overline{k}$-groups.

By discarding the action of E on D, we have a forgetful functor from the category of Galois extended $\overline{k}$-groups to the category of loosely Galois extended $\overline{k}$-groups. This functor is faithful, and by the above remarks it is fully faithful and injective on objects on the full subcategory of those Galois extended $\overline{k}$-groups (D,E) with D reduced.

Proposition 11.5. *Suppose that k is of characteristic 0. Then the forgetful functor from the category of Galois extended $\overline{k}$-groups to the category of loosely Galois extended $\overline{k}$-groups is an isomorphism of categories.*

Proof. Since any $\overline{k}$-group is reduced, it remains by the above remarks only to show that for any loosely Galois extended $\overline{k}$-group (D, E), the action $e \mapsto a_e$ defined above is continuous. Fix a section $s : \mathrm{Gal}(\overline{k}/k) \to E$ with $s(1) = 1$ of the continuous map $E \to \mathrm{Gal}(\overline{k}/k)$.

Suppose first that D is of finite type over $\overline{k}$. Then $D(\overline{k})_{\overline{k}}$ is discrete, and

$$D = D_0 \times_{k_0} \overline{k}$$

for a finite subextension k_0 of $\overline{k}$ and affine k_0-group of finite type D_0. By Lemma 11.2, $D(\overline{k})_{\overline{k}}$ has a finitely generated subgroup Φ which is dense in D. Increasing k_0 if necessary, we may suppose that a finite set of generators of Φ, and hence Φ itself, is contained in $D_0(k_0)_{k_0} \subset D(\overline{k})_{\overline{k}}$. Similarly we may suppose that conjugation by $s(\sigma)$ for σ in $\mathrm{Gal}(\overline{k}/k_0)$ fixes Φ. For σ in $\mathrm{Gal}(\overline{k}/k_0)$ we then have

$$a_{s(\sigma)} = D_0 \times_{k_0} \mathrm{Spec}(\sigma^{-1}),$$

because the two sides are σ-automorphisms of D which coincide on Φ. The restriction of the action of E on D to the inverse image E_0 of $\mathrm{Gal}(\overline{k}/k_0)$ in E is thus continuous, because it is given by an action

$$(E_0)_{/k} \times D \to (D(\overline{k})_{\overline{k}})_{/k} \times \mathrm{Gal}(\overline{k}/k_0)_{/k} \times D \to (D(\overline{k})_{\overline{k}})_{/k} \times D \to D$$

of $(E_0)_{/k}$, where the first arrow is defined by the continuous map that sends e above σ to $(es(\sigma)^{-1}, s(\sigma))$, the second by the action of $\mathrm{Gal}(\overline{k}/k_0)$ on D through its action on $\mathrm{Spec}(\overline{k})$, and the third by the action of $D(\overline{k})_{\overline{k}}$ on D by conjugation. Since E_0 is an open subgroup of E, the action of E on D is also continuous.

To prove the continuity for arbitrary D, we show that for every normal $\overline{k}$-subgroup N of D with D/N of finite type, there exists a normal $\overline{k}$-subgroup $N_0 \subset N$ of D with D/N_0 of finite type such that the closed subgroup $N_0(\overline{k})_{\overline{k}}$ of E is normal. Since $D(\overline{k})_{\overline{k}}/N_0(\overline{k})_{\overline{k}} = (D/N_0)(\overline{k})_{\overline{k}}$ by Lemma 11.1(i), we have for such an N_0 a loosely Galois extended $\overline{k}$-group $(D/N_0, E/N_0(\overline{k})_{\overline{k}})$. The action of E on D will then be the limit of continuous actions of the $E/N_0(\overline{k})_{\overline{k}}$ on D/N_0, and hence will be continuous.

The assignment $(\sigma, u) \mapsto s(\sigma).u.s(\sigma)^{-1}$ defines a continuous map

$$\mathrm{Gal}(\overline{k}/k) \times D(\overline{k})_{\overline{k}} \to D(\overline{k})_{\overline{k}}.$$

Since $N(\overline{k})_{\overline{k}}$ is an open subgroup of $D(\overline{k})_{\overline{k}}$, its inverse image under this map is open, and hence contains $J \times N'(\overline{k})_{\overline{k}}$ for some open normal subgroup J of $\mathrm{Gal}(\overline{k}/k)$ and normal $\overline{k}$-subgroup N' of D with D/N' of finite type. Then if $\sigma_1, \sigma_2, \ldots, \sigma_n$ are representatives for the cosets of J, we have

$$e.N(\overline{k})_{\overline{k}}.e^{-1} \supset \bigcap_{i=1}^{n} s(\sigma_i).N'(\overline{k})_{\overline{k}}.s(\sigma_i)^{-1}$$

for every e in E. Now

$$N'' = \bigcap_{i=1}^{n} a_{s(\sigma_i)}(N')$$

is a normal $\overline{k}$-subgroup of D with D/N'' of finite type, and by Lemma 11.1(ii) the inclusion $a_e(N) \supset N''$ holds for every e in E, because it holds on $\overline{k}$-points over $\overline{k}$. It thus suffices to take for N_0 the intersection of the $a_e(N)$ for e in E. $\square$

Let (D, E) be a Galois extended $\overline{k}$-group. By a *right* (D, E)*-scheme* we mean $\overline{k}$-scheme Z, together with a right action

$$c : Z \times_{\overline{k}} D \to Z$$

of the $\overline{k}$-group D on Z and a continuous right action of E on the underlying k-scheme of Z with e in E above σ in $\mathrm{Gal}(\overline{k}/k)$ acting as a σ^{-1}-automorphism b_e of Z, such that:

(a) $b_e \circ c = c \circ (b_e \times_{\mathrm{Spec}(\sigma)} a_{e^{-1}})$, where a_e is the action of e on D;

(b) $b_e = c(-, e)$ for e in $D(\overline{k})_{\overline{k}}$.

A morphism of right (D, E)-schemes is a morphism of $\overline{k}$-schemes which is compatible with the actions of D and E. For any k-scheme X we have a trivial right (D, E)-scheme $X_{\overline{k}}$, where the action of D is trivial and the action of E is that through the action of $\mathrm{Gal}(\overline{k}/k)$ on $\mathrm{Spec}(\overline{k})$.

If D is reduced and Z is a separated $\overline{k}$-scheme, then condition (a) in the definition of a right (D, E)-scheme is redundant. Indeed the equaliser of the two morphisms of (a) is a closed subscheme of $Z \times_{\overline{k}} D$, and hence coincides with it, because by (b) and condition (b) in the definition of a Galois extended $\overline{k}$-group, this equaliser contains the fibre at each point of the dense subset $D(\overline{k})_{\overline{k}}$ of D.

Just as a continuous action of E on the underlying k-scheme of a $\overline{k}$-group D for which e above σ acts as a σ-automorphism of D may be identified with an action (11.8) of the groupoid $E_{/\overline{k}}$ on D, a continuous right action of E on the underlying k-scheme of a $\overline{k}$-scheme Z for which e above σ acts as a σ^{-1}-automorphism of Z may be identified with a right action

$$\beta : Z \times_{\overline{k}} E_{/\overline{k}} \to Z \tag{11.14}$$

of $E_{/\overline{k}}$ on Z. Condition (a) in the definition of a right (D, E)-scheme is then equivalent the condition

$$\beta(c(z, d), w) = c(\beta(z, w), \alpha(w^{-1}, d)) \tag{11.15}$$

on points z, d and w of Z, D and $E_{/\overline{k}}$, and (b) to the condition

$$\beta(z, v) = c(z, \varepsilon(v)) \tag{11.16}$$

on points z of Z and v of $(D(\overline{k})_{\overline{k}})_{/\overline{k}}$, where ε is the counit (11.9). A morphism $Z \to Z'$ of right (D, E)-schemes is then a morphism $Z \to Z'$ of $\overline{k}$-schemes compatible with the actions of D and $E_{/\overline{k}}$.

Let F be a transitive affine groupoid over $\overline{k}$. Then if Z is a right F-scheme, we have a right action of F^{diag} on Z by restriction, and a right action of $F(\overline{k})_{\overline{k}}$ on Z with w acting as the $\gamma(w)^{-1}$-automorphism of Z defined by the action $Z \xrightarrow{\sim} \mathrm{Spec}(\gamma(w))^* Z$ at w of G on Z. The action of $F(\overline{k})_{\overline{k}}$ is continuous because it arises from an action of the form (11.14) with $E = F(\overline{k})_{\overline{k}}$, given by restricting the action of F along the counit (11.4). Further (11.15) and hence (a) is satisfied because the counit (11.4) preserves composition, and (11.16) and hence (b) is satisfied by naturality of the counit. Thus we obtain a functor from the category of right F-schemes to the category of right $(F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$-schemes.

Proposition 11.6. *Let F be a transitive affine groupoid over $\overline{k}$. Then passing from the action of F to the actions of F^{diag} and $F(\overline{k})_{\overline{k}}$ defines an isomorphism from the category of right F-schemes to the category of right $(F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$-schemes.*

Proof. By Lemma 11.3, the morphism $F \to \Gamma$ over $\overline{k}$ has a section $t : \Gamma \to F$, which we may suppose is the identity above the diagonal. It factors over $\overline{k}$ as $t_0 : \Gamma \to (F(\overline{k})_{\overline{k}})_{/\overline{k}}$ followed by the counit (11.4).

Let Z and Z' be right F-schemes. By the factorisation (11.5), a morphism $Z \to Z'$ of $\overline{k}$-schemes is compatible with the actions of F on Z and Z' provided it is compatible with the actions of F^{diag} and $(F(\overline{k})_{\overline{k}})_{/\overline{k}}$. This proves the full faithfulness.

Let Z be a right $(F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$-scheme. By the factorisation (11.5), a right G-scheme with underlying right $(F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$-scheme Z is unique if it exists, because for points d of F^{diag} and u of Γ with $d_0(u) = d_1(d)$, the point $d \circ t(u)$ of F must act as

$$z \mapsto \beta(c(z,d), t_0(u)),$$

where c is the action of F^{diag} and β is the action of $(F(\overline{k})_{\overline{k}})_{/\overline{k}}$. The action so defined is associative, because

$$\beta(c(z, t(u) \circ d \circ t(u)^{-1}), t_0(u)) = c(\beta(z, t_0(u)), d)$$

by (11.15), and

$$\beta(c(z, t(u) \circ t(u') \circ t(u \circ u')^{-1}), t_0(u \circ u')) = \beta(z, t_0(u) \circ t_0(u'))$$

by (11.16). It thus defines a structure of right F-scheme on Z. This proves the bijectivity on objects. □

Let X be a k-scheme and (D, E) be a Galois extended $\overline{k}$-group. Then we have a constant right (D, E)-scheme $X_{\overline{k}}$, where D acts trivially and E acts through the action of $\mathrm{Gal}(\overline{k}/k)$ on $\mathrm{Spec}(\overline{k})$. By a right (D, E)-scheme over X we mean a right (D, E)-scheme equipped with a morphism of right (D, E)-schemes to $X_{\overline{k}}$.

Let X be a non-empty k-scheme, H be a pregroupoid over X and (D, E) be a Galois extended $\overline{k}$-group. By a *principal* (H, D, E)*-bundle* we mean a right (D, E)-scheme P over X whose underling right D-scheme is a principal D-bundle over X, together with a structure of H-scheme on P whose defining isomorphism ${d_1}^*P \xrightarrow{\sim} {d_0}^*P$ is a (D, E)-morphism. A morphisn of (H, D, E)-bundles is a morphism of right (D, E)-schemes over X which is also an H-morphism. Such a morphism is necessarily an isomorphism.

Proposition 11.7. *Let X be a non-empty k-scheme, H be a pregroupoid over X, and F be a transitive affine groupoid over $\overline{k}$. Then passing from the action of F to the actions of F^{diag} and $F(\overline{k})_{\overline{k}}$ defines an isomorphism from the category of principal (H, F)-bundles to the category of principal $(H, F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$-bundles.*

Proof. Apply Proposition 11.6 to schemes over X and their pullbacks along d_0 and d_1. □

The existence and uniqueness up to unique isomorphism of a push forward of a principal (H, D, E)-bundle P along a morphism $(h, j) : (D, E) \to (D', E')$, i.e. a pair consisting of a principal (H, D', E')-bundle P' and a morphism $P \to P'$ compatible with h, j, and the actions of H, D, D', E and E', follows from Propositions 11.4 and 11.7, together with push forward for principal (H, F)-bundles. There is thus a functor from the category of Galois extended $\overline{k}$-groups to the category of sets which sends (D, E) to the set

$$H^1_H(X, D, E)$$

of isomorphism classes of principal (H, D, E)-bundles over X, and which is defined on morphisms using push forward. It factors through the category of Galois extended $\overline{k}$-groups up to conjugacy, where a morphism to (D, E) is a conjugacy class under the action of $D(\overline{k})_{\overline{k}}$. By Proposition 11.7, passing from the action of F to the actions of F^{diag} and $F(\overline{k})_{\overline{k}}$ defines a bijection

$$H^1_H(X, F) \xrightarrow{\sim} H^1_H(X, F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}}) \tag{11.17}$$

which is natural in the reduced transitive affine groupoid F over $\overline{k}$.

12. Levi subgroupoids and the fundamental cohomology class

In this section k is a field of characteristic 0 and X is a non-empty k-scheme.

For the remainder of this paper, we work over a field of characteristic 0.

In this section we determine the conditions under which the existence and uniqueness up to conjugacy of Levi subgroups of an affine k-group carry over to transitive affine groupoids. In general Levi subgroupoids of a transitive affine groupoid do not exist, but when they do exist any two of them are conjugate. It will be shown in Corollary 12.6 belows that the obstruction to the existence of a Levi subgroupoid is a suitably defined cohomology class.

Proposition 12.1. *If K is a reductive groupoid over X and U is a prounipotent K-group, then $H^1_K(X, U) = 1$.*

Proof. Let P be a (K, U)-torsor. It is to be shown that P has a section over X stabilised by K. Consider the set $\mathcal{P}$ of pairs (N, s) with N a normal K-group subscheme of U and s a section over X stabilised by K of the push forward P/N of P along the projection $U \to U/N$. Write $(N, s) \leq (N', s')$ when N contains N' and s is the image of s' under the projection from K/N' to K/N. Then $\mathcal{P}$ is inductively ordered, and hence has a maximal element (N, s).

Suppose that $N \neq 1$. Then for a sufficiently large K-quotient $\overline{U}$ of U of finite type, the image $\overline{N}$ of N under the projection $U \to \overline{U}$ is $\neq 1$. Write

$$1 = \overline{U}_n \subset \cdots \subset \overline{U}_1 \subset \overline{U}_0 = \overline{U}$$

for the lower central series of $\overline{U}$ and t for the largest i such that $\overline{U}_i$ contains $\overline{N}$. Then the intersection N' of N with the inverse image of $\overline{U}_{t+1}$ in U is a normal K-subgroup of U, and N/N' is of finite type and commutative. The inverse image Q of s under the the projection from P/N' to P/N is a $(K, N/N')$-torsor. Now

$$N/N' = \mathrm{Spec}(\mathrm{Sym}\, \mathcal{V})$$

for a representation $\mathcal{V}$ of K. Then $H^1_K(X, \mathcal{V}^\vee) = 0$ by semisimplicity of the category of K-modules, so that

$$H^1_K(X, N/N') = 1$$

Thus Q has a section s' over X stabilised by K. We may regard s' as a section of P/N' with image s in P/N. Then (N', s') is strictly greater than (N, s) in $\mathcal{P}$, contradicting the maximality of (N, s). Thus $N = 1$, and s is the required section of P. □

Let K be a reductive groupoid over X. By a *Levi subgroupoid* of K we mean a subgroupoid L of K such that the restriction to L of the projection from K onto K/R_uK is an isomorphism.

Lemma 12.2. *Any transitive affine groupoid over an algebraic extension of k has a Levi subgroupoid.*

Proof. Let K be a transitive affine groupoid over an algebraic extension k_1 of k. To prove that K has a Levi subgroupoid, suppose first that K is of finite type and that R_uK is commutative. Then we may suppose also that k_1 is finite over k. Let k' be a finite Galois extension of k which splits k_1 and is such that

$$K \to \mathrm{Spec}(k_1) \times \mathrm{Spec}(k_1)$$

is surjective on k'-points. Then $K_{k'}$ is a constant groupoid over the finite discrete k'-scheme $X_{k'}$. By the classical Levi decomposition, the set $\mathcal{L}$ of Levi subgroupoids of $K_{k'}$ is non-empty. The Galois group $\mathrm{Gal}(k'/k)$ acts on $\mathcal{L}$ by its action on $K_{k'}$ through k', and the fixed point set

$$\mathcal{L}^{\mathrm{Gal}(k'/k)}$$

is the set of Levi subgroupoids of K. Also the abelian group V of cross sections of $R_uK_{k'}$ acts by conjugation on $\mathcal{L}$. It follows easily from the constancy of $K_{k'}$ and the discreteness of $X_{k'}$ that V acts transitively on $\mathcal{L}$. Further V has a structure of k'-vector space, $\mathrm{Gal}(k'/k)$ acts k-linearly on V by its action on $R_uK_{k'}$ through k', and the action of V on $\mathcal{L}$ is compatible with the action of $\mathrm{Gal}(k'/k)$ on V and $\mathcal{L}$. Finally the stabiliser of $\mathcal{L}$ under V is the k'-vector subspace of V consisting of the cross sections of $R_uK_{k'}$ in the centre of $K_{k'}$. Thus $\mathcal{L}^{\mathrm{Gal}(k'/k)}$ is non-empty.

To prove the general case, consider the set $\mathcal{P}$ of pairs (N, L) with N a K-group subscheme of R_uK and L a Levi subgroupoid of K/N. Write $(K, L) \leq (K', L')$ when N contains N' and L is the image of L' under the projection from K/N' to K/N. Then $\mathcal{P}$ is inductively ordered, and hence has a maximal element (N, L).

Suppose that $N \neq 1$. Then N has a non-trivial K-quotient N/N' of finite type. We may assume, after passing if necessary to a smaller K-quotient, that N/N' is commutative. Write J for the inverse image of L under the the projection from K/N' to K/N. Then $R_uJ = N/N'$. For a sufficiently large K-quotient J_1 of J of finite type, the projection from J to J_1 induces an isomorphism from R_uJ to R_uJ_1. By what has already been shown, J_1 has a Levi subgroupoid. Its inverse image L' under the projection from J to J_1 is then a Levi subgroupoid of J, and hence also of K/N'. Then (N', L') is strictly greater than (N, L) in $\mathcal{P}$, contradicting the maximality of (N, L). Thus $N = 1$, and L is a Levi subgroupoid of K. □

Proposition 12.3. *Let K and K' be transitive affine groupoids over X, with K' reductive. Then any two morphisms from K' to K whose composites with the projection from K to K/R_uK coincide are conjugate by a section of R_uK over X.*

Proof. Write p for the projection from K to K/R_uK. Let f_1 and f_2 be morphisms from K' to K with $p \circ f_1 = p \circ f_2$. The we have a structure of K'-scheme on R_uK where the point z of K' sends the point u of R_uK to

$$f_2(z) \circ u \circ f_1(z)^{-1}.$$

The k-scheme U_{f_1,f_2} of sections of R_uK stabilised by K' exists and is affine over k, as is seen by reducing to the case where K' is constant. Its k-points are the sections α of R_uK with $f_2 = \mathrm{int}(\alpha) \circ f_1$. Given also $f_3 : K' \to K$ with $p \circ f_2 = p \circ f_3$, the composition of R_uK defines a k-morphism

$$U_{f_2,f_3} \times U_{f_1,f_2} \to U_{f_1,f_3},$$

and such k-morphisms have the evident composition and unit properties. We show that the k-scheme U_{f_1,f_2} is non-empty. It will then be a principal bundle under the prounipotent k-group U_{f_1,f_1}, and hence by Proposition 12.1 with $X = \mathrm{Spec}(k)$ and $K = 1$ have a k-point, as required.

We may write K as the filtered limit of its quotients K_λ of finite type. Then R_uK is the limit of the R_uK_λ. Since a filtered limit of non-empty affine schemes is non-empty and since formation of the U_{f_1,f_2} commute with limits, we may suppose that K is of finite type. Both f_1 and f_2 then factor through a quotient of K' of finite type, so that we may suppose K' is also of finite type. By compatibility of U_{f_1,f_2} with extension of scalars, we may suppose further that X has a k-point. Pulling back along the inclusion of such a k-point and using the compatiblity of U_{f_1,f_2} with pullback, we may suppose finally that $X = \mathrm{Spec}(k)$. Then K and K' are affine k-groups of finite type. The pullback $r : L \to K'$ of p along

$$p \circ f_1 = p \circ f_2 : K' \to K/R_uK$$

has kernel the unipotent k-group R_uK. Since K' is reductive, the splittings of r defined by f_1 and f_2 are conjugate by a k-point of R_uK, by the classical Levi decomposition for L. Thus f_1 and f_2 are themselves conjugate by a k-point of R_uK. It follows that U_{f_1,f_2} has a k-point, and hence is non-empty. □

Lemma 12.4. *Let K be a transitive affine groupoid over X. Then there exists an fpqc covering morphism $X' \to X$ such that the pullback of K along $X' \to X$ has a Levi subgroupoid.*

Proof. By Lemma 7.4, we may suppose that $K = \underline{\mathrm{Iso}}_G(P)$ for a transitive affine groupoid G over $\mathrm{Spec}(\overline{k})$. By Lemma 12.2, G has a Levi subgoupoid. Hence by (6.3), the pullback of K along $P \to X$ has a Levi subgroupoid. □

Let K be a transitive affine groupoid over X. If $K \to K'$ is a faithfully flat morphism of groupoids over X and U is a K-subgroup of K^{diag} with image U' in K'^{diag}, then K acts on the push forward P' of a (K,U)-torsor P along $U \to U'$ through K'. Thus by assigning to the class of P the class P' we obtain a map

$$H^1_K(X,U) \to H^1_{K'}(X,U') \tag{12.1}$$

of pointed sets. The map (12.1) is compatible with pullback of K and K' along a morphism of K-schemes $X' \to X$,

The image of R_uK under any faithfully flat morphism $K \to K'$ of groupoids over X is R_uK'. Such a morphism thus induces a map

$$H^1_K(X,R_uK) \to H^1_{K'}(X,R_uK') \tag{12.2}$$

of pointed sets. Again it is compatible with pullback.

To give a semidirect product decomposition $U \rtimes L$ of K is to give an idempotent endomorphism of K with kernel U. If e is such an endomorphism, define a (K,U)-torsor U^e as follows. The underlying scheme over X of U^e is U. The action of K on U^e is that where the point v of K with source x_1 and target x_0 sends the point u of U above x_1 to

$$e(v) \circ u \circ v^{-1}$$

above x_0. The right action of the K-scheme U on U^e is given by the composition of U. We write

$$\varphi_{K,e} \in H^1_K(X,U)$$

for the class of U^e. If u is a cross section of U and $e' = \mathrm{int}(u) \circ e$ is the conjugate of e by u, then

$$\varphi_{K,e} = \varphi_{K,e'}, \tag{12.3}$$

because left translation by u defines an isomorphism $U^e \xrightarrow{\sim} U^{e'}$ of (K,U)-torsors.

Let $h : K \to K'$ be a faithfully flat morphism of groupoids over X and e' be the idempotent endomorphism of K' corresponding to the direct sum decomposition $U' \rtimes L'$ of K'. Then $h \circ e = e' \circ h$ if and only if U' is the image of U and L' is the image of L under h. When this is so, the map (12.1) induced by h sends $\varphi_{K,e}$ to $\varphi_{K',e'}$.

We define as follows the *fundamental cohomology class*

$$\varphi_K \in H^1_K(X, R_u K)$$

of K. Suppose first that K has a Levi subgroupoid L. Then if e is the idempotent endomorphism of K defined by L, we set

$$\varphi_K = \varphi_{K,e}.$$

That φ_K so defined is independent of the choice of L follows from Proposition 12.3 and (12.3). In general, there exists by Lemma 12.4 an *fpqc* covering morphisms $X' \to X$ along which the pullback K' of K has a Levi subgroup. Then we define φ_K by requiring that its image under the pullback bijection be $\varphi_{K'}$. The independence of the choice of $X' \to X$ follows from the fact that for any two such choices, their fibre product over X is a third which factors through both.

Clearly the fundamental cohomology class is preserved by arbitrary pullback, and by extension of the scalars. It is also preserved by the map (12.2) induced by a faithfully flat $K \to K'$, because (12.2) is compatible with pullback.

Proposition 12.5. *Let H be a pregroupoid over X, K be a transitive affine groupoid over K, and P be a (K, R_uK)-torsor with class in $H^1_K(X, R_uK)$ the fundamental class. Then the stabiliser under K of any element of $H^0_H(X,P)$ is a Levi subgroupoid of K over H, and every Levi subgroupoid of K over H is the stabiliser of some element of $H^0_H(X,P)$.*

Proof. The stabiliser under K of any element of $H^0_H(X,P)$ is a subgroupoid over H. We may thus suppose that a Levi subgroupoid L of K exists: for the first statement we reduce to this case by Lemma 12.4, and the second statement is empty unless an L exists.

Write e for the idempotent endomorphism of K corresponding to L. Then the (K, R_uK)-torsors P and $(R_uK)^e$ are isomorphic, and we may suppose that

$$P = (R_uK)^e.$$

The stabiliser K_u of an element u of $H^0(X,P)$ is then the conjugate of L by u^{-1}, and hence a Levi subgroupoid of K. If L is a subgroupoid over H and u is the identity section of P, then u lies in $H^0_H(X,P)$ and $K_u = L$. □

Corollary 12.6. *Let H be a pregroupoid over X and K be a transitive affine groupoid over K. Then K has a Levi subgroupoid over H if and only if the fundamental class in $H^1_K(X, R_uK)$ has image the base point in $H^1_H(X, R_uK)$. When this is so, any two Levi subgroupoids of K over H are conjugate by an element of $H^0_H(X, R_uK)$.*

Proof. Immediate from the Proposition. □

The conjugacy statement of the above corollary can also be deduced directly from Proposition 12.3.

13. REDUCTIVE SUBGROUPOIDS

In this section k is a field of characteristic 0 and X is a non-empty k-scheme.

The main result of this section is Theorem 13.4, which gives necessary and sufficient conditions for a transitive affine groupoid to contain a reductive subgroupoid. It follows in particular from this result that any reductive subgroupoid of a transitive affine groupoid is contained in a Levi subgroupoid. This will be used later in showing that results valid for reductive groupoids can sometimes be extended to more general transitive affine groupoids.

Recall that if G is a k-group of finite type and G' is a subgroup of G, then the quotient G/G' exists.

Lemma 13.1. *Let G be a k-group of finite type, G' be a k-subgroup of G, and G_0 be a normal k-subgroup of G. Denote by G_1 the k-quotient G/G_0 and by G'_1 the image of G' under the projection $G \to G_1$. Suppose that $G_0/(G_0 \cap G')$ and G_1/G'_1 are affine. Then G/G' is affine.*

Proof. We have a cartesian square of k-schemes

$$\begin{array}{ccc} G/G' & \longleftarrow & G \times (G_0/(G_0 \cap G')) \\ \big\downarrow & & \big\downarrow \\ G/(G_0 G') & \longleftarrow & G \end{array}$$

where the top arrow is the morphism of G-schemes obtained from the morphism of k-schemes $G_0/(G_0 \cap G') \to G/G'$ induced by the embedding $G_0 \to G$, the left and bottom arrow are the projections, and the right arrow is the first projection. The left arrow is affine because the right arrow is affine and the bottom arrow is faithfully flat. Since the projection $G \to G_1$ induces an isomorphism from $G/(G_0 G')$ to G_1/G'_1, it follows that G/G' is affine. $\square$

Lemma 13.2. *Let G be a k-group of finite type and G' be a k-subgroup of G with $R_u G'$ contained in $R_u G$. Then G/G' is affine.*

Proof. It is enough to show that $R_u G/(R_u G \cap G')$ is affine, because by Matsushima's criterion, the hypotheses of Lemma 13.1 are then satisfied with $G_0 = R_u G$. Thus we reduce to the case where G is unipotent. In that case we may argue by induction on the dimension of G. Indeed if G is of dimension > 0, then the hypotheses of Lemma 13.1 are satisfied with G_0 the derived group of G, because G_0 is of dimension strictly less than that of G while G_1 is commutative. $\square$

Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories, $F : \mathcal{A} \to \mathcal{A}'$ be an exact functor, and $F' : \mathcal{A}' \to \mathcal{A}$ be a right adjoint to F. Then there is a canonical homomorphism

$$\operatorname{Ext}^1_{\mathcal{A}}(M, F'(N)) \to \operatorname{Ext}^1_{\mathcal{A}'}(F(M), N) \tag{13.1}$$

given by applying F to an extension of M by $F'(N)$ and then pushing forward along the counit $FF'(N) \to N$. Suppose that F' is also exact. Then (13.1) is an isomorphism, with inverse given by applying F' to an extension of $F(M)$ by N and then pulling back along the unit $M \to F'F(M)$.

Lemma 13.3. *Let K be a transitive affine groupoid over X and K' be a transitive affine subgroupoid of K with $R_u K'$ contained in $R_u K$. Then for every $\mathcal{V}'$ in $\mathrm{Mod}_{K'}(X)$ and v' in $H^1_{K'}(X, \mathcal{V}')$ there exists a $\mathcal{V}$ in $\mathrm{Mod}_K(X)$, a v in $H^1_K(X, \mathcal{V})$ and a morphism f from $\mathcal{V}$ to $\mathcal{V}'$ in $\mathrm{Mod}_{K'}(X)$, such that f sends v to v'.*

Proof. K acts on $\mathcal{V}$ through a quotient K_1 of finite type. Since inflation defines injective homomorphisms on the groups H^1, we may after replacing K by K_1 and K' by its image in K_1 suppose that K is of finite type. By Lemma 13.2, the K-scheme K/K' exists and is affine over X. Write

$$K/K' = Y = \mathrm{Spec}(\mathcal{R})$$

for a commutative K-algebra $\mathcal{R}$.

The embedding of K' into K factors as

$$(X, K') \to (Y, K \times_X Y) \to (X, K),$$

in the category of groupoids in k-schemes, where the first arrow is the embedding of the pullback along the base cross-section of Y and the second arrow is given by the first projection. If we write $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ for $\mathrm{MOD}_K(X)$, $\mathrm{MOD}_{K'}(X)$ and $\mathrm{MOD}_{K \times_X Y}(Y)$, then the restriction functor F from $\mathcal{A}$ to $\mathcal{A}'$ factors as a functor $\mathcal{A} \to \mathcal{A}''$ induced by the second arrow followed by an equivalence $\mathcal{A}'' \to \mathcal{A}'$ induced by the first. Further $\mathcal{A}''$ is equivalent to the category of $\mathcal{R}$-modules in $\mathcal{A}$, by (8.6). Modulo this equivalence, $\mathcal{A} \to \mathcal{A}''$ is the functor $\mathcal{R} \otimes -$, with exact right adjoint the forgetful functor from $\mathcal{R}$-modules in $\mathcal{A}$ to $\mathcal{A}$. Thus F has an exact right adjoint F'.

Since both F and F' are exact, (13.1) with $M = \mathcal{O}_X$ and $N = \mathcal{V}'$ shows that restricting to K' and pushing forward along the counit $FF'(\mathcal{V}') \to \mathcal{V}'$ gives an isomorphism

$$H^1_K(X, F'(\mathcal{V}')) = \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{O}_X, F'(\mathcal{V}')) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{A}'}(\mathcal{O}_X, \mathcal{V}') = H^1_{K'}(X, \mathcal{V}').$$

Let $\mathcal{E}$ be an extension of $\mathcal{O}_X$ by $F'(\mathcal{V}')$ in $\mathcal{A}$ whose class corresponds under this isomorphism to v'. Writing $\mathcal{E}$ as the filtered colimit of its K-submodules in $\mathrm{Mod}_K(X)$ shows that there is such a K-submodule $\mathcal{E}_0$ such that the restriction to $\mathcal{E}_0$ of $\mathcal{E} \to \mathcal{O}_X$ is non-zero, and hence an epimorphism. The kernel $\mathcal{V}$ of $\mathcal{E}_0 \to \mathcal{O}_X$ is a K-submodule of $F'(\mathcal{V}')$ which lies in in $\mathrm{Mod}_K(X)$, and the embedding sends the class v of $\mathcal{E}_0$ to the class of $\mathcal{E}$. Thus $\mathcal{V}$ and v have the required property. □

Theorem 13.4. *Let H be a pregroupoid over X and K be a transitive affine groupoid over H. Then the following conditions are equivalent.*

(a) *K has a reductive subgroupoid over H.*
(b) *K has a Levi subgroupoid over H.*
(c) *The morphism of representations of H underlying any non-zero morphism of representations K with target $\mathcal{O}_X$ is a retraction.*

Proof. (a) $\Longrightarrow$ (c): Let K' be a reductive subgroupoid of K over H. Since any non-zero morphism of representations of K with target $\mathcal{O}_X$ is surjective, it is a retraction as a morphism of representations of K', and hence as a morphism of representations of H.

(c) $\Longrightarrow$ (b): Suppose that (c) holds. Let K' be a transitive affine subgroupoid of K over H such that $R_u K'$ is contained in $R_u K$. We first show that for any representation $\mathcal{V}'$ of K' the homomorphism

$$H^1_{K'}(X, \mathcal{V}') \to H^1_H(X, \mathcal{V}') \tag{13.2}$$

induced by the H structure on K' is 0. Given v' in $H^1_{K'}(X, \mathcal{V})$, choose a $\mathcal{V}$, v and $f : \mathcal{V} \to \mathcal{V}'$ as in Lemma 13.3. By (c) and the identification of the H^1 with groups of extensions, the image of v in $H^1_H(X, \mathcal{V})$, and hence in $H^1_H(X, \mathcal{V}')$, is 0. Thus the image of v' in $H^1_H(X, \mathcal{V}')$ is 0.

Consider now the set $\mathcal{P}$ of pairs (M, L) with M a K-quotient of K by a K-subgroup of R_uK and L a Levi subgroupoid of M over H. Write $(M, L) \leq (M', L')$ when M factors through M' and L is the image of L' under the projection from M' onto M. Then $\mathcal{P}$ is inductively ordered, and hence has a maximal element. Let (M_0, L_0) be an element of $\mathcal{P}$ with $K \neq M_0$. We show that there is an (M, L) in $\mathcal{P}$ strictly greater than (M_0, L_0). Then (b) will follow, because (M_1, L_1) maximal in $\mathcal{P}$ will imply $K = M_1$.

By hypothesis M_0 is the quotient of K by a K-subgroup $N_0 \neq 1$ of R_uK. For a sufficiently large K-subgroup $N \neq N_0$ of N_0, the quotient N_0/N is of finite type and commutative. Write M for the quotient of K by N and J for the inverse image of L_0 in M. Then $R_uJ = N_0/N$. The homomorphism

$$H^1_J(X, R_uJ) \to H^1_H(X, R_uJ)$$

induced by the H-structure on J is 0, because it factors through (13.2) with K' the inverse image of J in K and $\mathcal{V}'$ the representation of K' underlying the representation of K such that

$$R_uJ = N_0/N = \mathrm{Spec}(\mathrm{Sym}\, \mathcal{V}'^{\vee}).$$

Thus by Corollary 12.6, J has a Levi subgroupoid L over H. Then L is also a Levi subgroupoid of M over H, and (M, L) is strictly greater than (M_0, L_0) in $\mathcal{P}$.

(b) $\implies$ (a): Immediate. □

Note that if K is a transitive affine groupoid over H, it follows from Proposition 12.3 that any two Levi subgroupoids of K over H are conjugate by an element of $H^0_H(X, R_uK)$.

Theorem 13.5. *Let H be a pregroupoid over X, K be a transitive affine groupoid over H, and k' be an extension of k. Then K has a reductive subgroupoid over H if and only if $K_{k'}$ has a reductive subgroupoid over $H_{k'}$.*

Proof. That $K_{k'}$ has a reductive subgroupoid over $H_{k'}$ if K has a reductive subgroupoid over H is immediate.

Suppose conversely that $K_{k'}$ has a reductive subgroupoid over $H_{k'}$. Let $f : \mathcal{V} \to \mathcal{O}_X$ be a non-zero morphism of representations of K. Then by Theorem 13.4, there exists a morphism $g' : \mathcal{O}_{X_{k'}} \to \mathcal{V}_{k'}$ of representations of $H_{k'}$ right inverse to $f_{k'}$. We may regard $f_{k'}$ and g' as morphisms

$$\mathcal{O}_X \otimes_k k' \xrightarrow{g'} \mathcal{V} \otimes_k k' \xrightarrow{f \otimes_k k'} \mathcal{O}_X \otimes_k k'$$

in $\mathrm{MOD}_H(X)$ with composite the identity, where H acts trivially on k'. Let $r : k' \to k$ be a homomorphism of k-vector spaces left inverse to the embedding $e : k \to k'$. Then the composite

$$\mathcal{O}_X \xrightarrow{\mathcal{O}_X \otimes e} \mathcal{O}_X \otimes_k k' \xrightarrow{g'} \mathcal{V} \otimes_k k' \xrightarrow{\mathcal{V} \otimes_k r} \mathcal{V}$$

is right inverse to f. Thus K is reductive over H, by Theorem 13.4. □

The above results can be applied to principal bundles using the following theorem.

Theorem 13.6. *Let H be a pregroupoid over X, G be an affine k-group, and P be a principal (H,G)-bundle. Then $\underline{\mathrm{Iso}}_G(P)$ has a reductive subgroupoid over H if and only if there exists a reductive k-subgroup G' of G for which P has a principal (H,G')-subbundle.*

Proof. If G' has a reductive k-subgroup of G for which P has a principal (H,G')-subbundle P', then $\underline{\mathrm{Iso}}_{G'}(P')$ is reductive and the embeddings of G' into G and P' into P define an embedding of $\underline{\mathrm{Iso}}_{G'}(P')$ into $\underline{\mathrm{Iso}}_G(P)$ over H.

Conversely suppose that $\underline{\mathrm{Iso}}_G(P)$ has a reductive subgroupoid over H. Write $\overline{G}$ for G/R_uG and $\overline{q} : P \to \overline{P}$ for the push forward morphism along the projection $\overline{h} : G \to \overline{G}$. We may identify $\underline{\mathrm{Iso}}_{\overline{h}}(\overline{q})$ with the projection from $\underline{\mathrm{Iso}}_G(P)$ onto its quotient by its unipotent radical. By Theorem 13.4, $\underline{\mathrm{Iso}}_G(P)$ has a Levi subgroupoid over H, so ttat there is thus a morphism e over H right inverse to $\underline{\mathrm{Iso}}_{\overline{h}}(\overline{q})$. By Lemma 5.2 and the naturality of (5.4) in (G',P'), the obstruction in $H^1(k,G)$ to the existence of an (h,q) with $e = \underline{\mathrm{Iso}}_h(q)$ has image in $H^1(k,\overline{G})$ the base point, and is thus itself the base point by Proposition 12.1 and the exact cohomology sequence. Hence $e = \underline{\mathrm{Iso}}_h(q)$ for some (h,q). The required G' and principal (H,G')-subbundle are given by the images of h and q. □

14. The splitting theorem

In this section k is a field of characteristic 0.

In this section we prove the splitting theorem in the form that will be required. This will be done by suitably adapting the the proof of the form given in [O'S11]. The main idea there was to reduce a categorical problem about tensor categories to a geometrical problem about actions of reductive groups on affine schemes. Here the reduction to a geometrical problem applies almost unmodified, and and it is only necessary to adapt the proof of [O'S11, Lemma 4.4.4] to give the more general Lemma 14.3 below. It will be convenient to include in this section some results from the Theory of Tannakian categories that will be reqired.

We begin by recalling some definitions and properties of affine k-groups and their modules. Many of these are particular cases of those of transitive affine groupoids, which over k reduce to affine k-groups. We note that what are here called reductive k-groups (resp. reductive k-groups of finite type) were called proreductive k-groups (resp. reductive k-groups) in [O'S11].

Let G be an affine k-group. By a G-module we mean a k-vector space V equipped with an action of G. When V is finite-dimensional we also speak of a representation of V. The category of G-modules is abelian, and has limits and colimits. The forgetful functor from G modules to k-vector spaces preserves colimits and finite limits but does not in general preserve arbitrary limits: the canonical homomorphism from the underling k-vector space of a limit of G-modules to the limit of the underlying k-vector spaces is injective but not in general surjective. Every G-module is the filtered colimit of its finite dimensional G-submodules. If V is a finite-dimensional G-module, then $\mathrm{Hom}_G(V,-)$ preserves filtered colimits.

The tensor product over k of G-modules has a canonical structure of G-module, and the category of G-modules then forms a k-tensor category. Let V be a finite-dimensional G-module. Then the dual $V^\vee$ of V has a canonical structure of G-module. For any G-module W, there is then a canonical isomorphism

$$\mathrm{Hom}_G(V,W) \xrightarrow{\sim} (V^\vee \otimes W)^G \tag{14.1}$$

which is natural in V and W.

By a G-algebra we mean a G-module R equipped with a structure of k-algebra for which the multiplication $R \otimes_k R \to R$ and identity $k \to R$ are G-homomorphisms. Given a G-algebra R, a (G, R)-module is a module M over the k-algebra R such that the action $R \otimes_k M \to M$ of R on M is a G-homomorphism. The forgetful functor from G-algebras to G-modules preserves limits and filtered colimits, and similarly for the forgetful functor from (G, R)-modules to G-modules.

A G-algebra will be called finitely generated if it is finitely generated as a k-algebra. A finitely generated G-algebra has a finite-dimensional G-submodule which generates it over k, because any finite set of generators is contained in such a G-submodule. A (G, R)-module will be called finitely generated if it is finitely generated over R. A finitely generated (G, R)-module has a finite-dimensional G-submodule which generates it over R.

Let R be a finitely generated commutative G-algebra. Then $\mathrm{Hom}_{G-\mathrm{alg}}(R, -)$ preserves filtered colimits of commutative G-algebras. This is clear when R is the symmetric algebra $\mathrm{Sym}\, V$ on a finite-dimensional G-module V, because then $\mathrm{Hom}_{G-\mathrm{alg}}(R, -)$ is naturally isomorphic to $\mathrm{Hom}_G(V, -)$. In general, there are finite-dimensional G-modules V and V' such that R is the coequaliser in the category of commutative G-algebras of two morphisms from $\mathrm{Sym}\, V'$ to $\mathrm{Sym}\, V$, and it suffices to note that in the category of sets finite limits commute with filtered colimits.

An affine k-group G is reductive if and only if the category of G-modules is semisimple (i.e. every short exact sequence splits). If G is reductive, every G-module is thus projective and injective. In particular the trivial G-module k is projective, so that the functor $(-)^G$ that assigns to a G-module its space of invariants is exact.

Lemma 14.1. *Let G be an affine k-group and W be a G-module. Suppose that k is algebraically closed. Then every $G(k)$-subspace of W is a G-submodule.*

Proof. A k-subspace of W is a G-submodule provided that its intersection with every finite-dimensional G-submodule W' of W is. We may thus suppose that $W = W'$ is finite dimensional, and that G is of finite type. Let W_0 be a $G(k)$-subspace of W. Then if W_0 and W are regarded as k-schemes, the restriction to $G \times W_0$ of the action morphism $G \times W \to W$ factors through W_0, because it does so on k-points and $G \times W$ is reduced. Thus W_0 is a G-submodule of W. □

Let G be a reductive k-group of finite type and R be a finitely generated commutative G-algebra. Then the R^G is a finitely generated k-algebra (e.g. [Sha94, II Theorem 3.6]). Let M be a finitely generated (G, R)-module. Then M^G is a finitely generated R^G-module (e.g. [Sha94, II Theorem 3.25]). It follows that $\mathrm{Hom}_G(V, M)$ is a finitely generated R^G-module for any finite-dimensional G-module V, because (14.1) with $W = M$ is an isomorphism of R^G-modules.

Let G be a reductive k-group and R be a commutative G-algebra for which R^G is a local k-algebra. Then R has a unique maximal G-ideal. Indeed if $\mathcal{J}$ is the set of G-ideals $J \neq R$ of R, then applying the exact functor $(-)^G$ shows that the image of the canonical G-homomorphism from $\coprod_{J \in \mathcal{J}} J$ to R does not contain 1.

For the proof of Lemma 14.2 below we need the following fact from commutative algebra (e.g. [Bou85, III, §3 Proposition 5 and IV, §1 Proposition 2, Corollaire 2]): given an ideal J in a noetherian commutative ring R and a finitely generated

R-module M, we have

$$\bigcap_{n=1}^{\infty} J^r M = 0 \tag{14.2}$$

if and only if $J + \mathfrak{p} \neq R$ for every associated prime $\mathfrak{p}$ of M.

Lemma 14.2. *Let G be a reductive k-group of finite type, R be a G-algebra, $J \neq R$ be a G-ideal of R, and M be a finitely generated (G, R)-module. Suppose that R^G is a complete noetherian local k-algebra with residue field k, and that R is finitely generated as an algebra over R^G. Then M is the limit in the category of G-modules of its quotients $M/J^n M$.*

Proof. We note that the conclusion of the lemma will hold provided that for every finite-dimensional G-module V the canonical homomorphism

$$\operatorname{Hom}_G(V, M) \to \lim_n \operatorname{Hom}_G(V, M/J^n M) \tag{14.3}$$

is bijective. Indeed (14.3) will then be bijective for arbitrary V, as follows by writing V as the filtered colimit of its finite-dimensional G-submodules. We write $\mathfrak{m}$ for the radical of R^G. The proof proceeds in three steps.

(1) Suppose that R^G is is finite over k. Let V be a finite-dimensional G-module. Then $\operatorname{Hom}_G(V, M)$ is finite-dimensional over k because it is a finitely generated R^G-module. Hence (14.3) is surjective because $\operatorname{Hom}_G(V, -)$ is exact. To see that (14.3) is injective, it suffices to check that (14.2) holds. To do this we may after extending the scalars suppose that k is algebraically closed. Let $\mathfrak{p}$ be an associated prime of M. Then

$$\mathfrak{p}_0 = \bigcap_{g \in G(k)} g\mathfrak{p}$$

is stable under $G(k)$, and hence by Lemma 14.1 is a G-ideal of R. Since R^G is local,

$$R^G \to (R/J)^G \times (R/\mathfrak{p}_0)^G$$

is not surjective. Thus $R \to R/J \times R/\mathfrak{p}_0$ is not surjective, so that

$$J + \mathfrak{p}_0 \neq R.$$

Since each $g\mathfrak{p}$ lies in the finite set of associated primes of M, we therefore have $J + g\mathfrak{p} \neq R$ for some $g \in G(k)$. Thus

$$J + \mathfrak{p} = g^{-1}(J + g\mathfrak{p}) \neq R.$$

Hence (14.2) holds as required.

(2) Suppose that $J = \mathfrak{m}R$. Let V be a finite-dimensional G-module. Since $\operatorname{Hom}_G(V, -)$ preserves the cokernel of

$$\mathfrak{m}^n \otimes_k M \to M,$$

there is for each n a canonical isomorphism

$$\operatorname{Hom}_G(V, M)/\mathfrak{m}^n \operatorname{Hom}_G(V, M) \xrightarrow{\sim} \operatorname{Hom}_G(V, M/\mathfrak{m}^n M).$$

Thus (14.3) is bijective, because $\operatorname{Hom}_G(V, M)$ is a finitely generated R^G-module and hence complete for the $\mathfrak{m}$-adic topology.

(3) Now consider the general case. Write R_r and M_r for $R/\mathfrak{m}^r R$ and $M/\mathfrak{m}^r M$, and J_r for the image of J in R_r. Then $J_r \neq R_r$, because J and $\mathfrak{m}^r R$ are contained

in the unique maximal G-ideal of R. By (2), and by (1) with R_r for R, J_r for J and M_r for M, we have canonical isomorphisms of G-modules

$$M \xrightarrow{\sim} \lim_r M_r \xrightarrow{\sim} \lim_r \lim_n M_r/(J_r)^n M_r. \tag{14.4}$$

Also $M_{(n)} = M/J^n M$ is a finitely generated R-module for each n, so that by (2) with $M_{(n)}$ for M, the second arrow of

$$M \to \lim_n M_{(n)} \xrightarrow{\sim} \lim_n \lim_r M_{(n)}/\mathfrak{m}^r M_{(n)} \tag{14.5}$$

is an isomorphism. On the other hand, by (14.4) and the canonical isomorphisms

$$M_r/(J_r)^n M_r \xleftarrow{\sim} M/(J^n M + \mathfrak{m}^r M) \xrightarrow{\sim} M_{(n)}/\mathfrak{m}^r M_{(n)},$$

the composite (14.5) is an isomorphism. Thus $M \to \lim_n M_{(n)}$ is an isomorphism, as required. □

Let G be an affine k-group. A commutative G-algebra R will be called *simple* if $R \neq 0$ and R has no G-ideal other than 0 and R. Suppose that G is of finite type and R is a simple commutative G-algebra with $R^G = k$. Then by a theorem of Magid [Mag87, Theorem 4.5], $\mathrm{Spec}(R)$ is a homogeneous space under G. In particular, R is a finitely generated k-algebra.

Let A be the henselisation at a k-point of k-scheme of finite type, and F be a functor from commutative A-algebras to sets. Then if $\widehat{A}$ is the completion of A, it follows from Artin's approximation theorem [Art69, Theorem 1.12] that $F(A)$ is non-empty provided that $F(\widehat{A})$ is non-empty and F preserves filtered colimits.

The following lemma reduces when $R^G = k$ to [O'S11, Lemma 4.4.4]. Since some of the steps in the proof of [O'S11, Lemma 4.4.4] apply essentially unmodified to in the proof of Lemma 14.3, we simply refer at the relevant places to [O'S11].

Lemma 14.3. *Let G be a reductive k-group, R be a commutative G-algebra, D be a G-subalgebra of R, and $p : R \to \overline{R}$ be the projection onto a simple quotient G-algebra of R. Suppose that R^G is a henselian local k-algebra with residue field k, and that the restriction of p to D is injective. Then R has a G-subalgebra D' containing D such that the restriction of p to D' is an isomorphism.*

Proof. We have $\overline{R}^G = k$, because $R^G \to \overline{R}^G$ is surjective, R^G has residue field k, and $\overline{R}$ is G-simple. Thus by Magid's theorem, $\overline{R}$ will be a finitely generated G-algebra provided that G is of finite type.

Write J for the G-ideal $\mathrm{Ker}\, p$ of R and A for the k-algebra R^G. Since $\overline{R} \neq 0$, we have $J \neq R$. We consider successively the following cases:

(1) G is of finite type, $J^2 = 0$, and D is a finitely generated k-algebra
(2) G is of finite type, A is a complete noetherian local k-algebra, R is a finitely generated A-algebra, and D is a finitely generated k-algebra
(3) G is of finite type, A is the henselisation at a k-point of a k-scheme of finite type, R is a finitely generated A-algebra, and D is a finitely generated k-algebra
(4) G is of finite type and D is a finitely generated k-algebra
(5) the general case.

When $A = k$, the cases (1), (4) and (5) above are respectively the cases (1), (2) and (3) of [O'S11, Lemma 4.4.4].

(1) Step (1) of the proof of [O'S11, Lemma 4.4.4] applies unmodified.

(2) By Lemma 14.2, R is the limit in the category of G-algebras of its quotients R/J^n. If D_n is the image of D in R/J^n, it is thus enough to show that any G-subalgebra $D'_n \supset D_n$ of R/J^n with $D'_n \to \overline{R}$ an isomorphism can be lifted to a G-subalgebra $D'_{n+1} \supset D_{n+1}$ of R/J^{n+1} with $D'_{n+1} \to \overline{R}$ an isomorphism. To do this, apply (1) with the inverse image of D'_n in R/J^{n+1} for R and D_{n+1} for D.

(3) Given a commutative A-algebra B, denote by $F(B)$ the set of those G-algebra homomorphisms

$$a : \overline{R} \to R \otimes_A B$$

such that the restrictions to D of $a \circ p$ and the canonical homomorphism $R \to R \otimes_A B$ coincide, and such that $(p \otimes_A B) \circ a$ is the canonical homomorphism $\overline{R} \to \overline{R} \otimes_A B$. Then $B \mapsto F(B)$ may be regarded as a functor from commutative A-algebras to sets. It is enough to show that $F(A)$ is non-empty, because with the identification $R = R \otimes_A A$ we may take $D' = a(\overline{R})$ for any a in $F(A)$. By Artin's approximation theorem, it will suffice to show that F commutes with filtered colimits and that $F(\widehat{A})$ is non-empty, where $\widehat{A}$ is the completion of A.

Since $\overline{R}$ is a finitely generated G-algebra, $\mathrm{Hom}_{G-\mathrm{alg}}(\overline{R}, -)$ preserves filtered colimits of commutative G-algebras. Thus if B is the filtered colimit of commutative A-algebras $\mathrm{colim}_\lambda B_\lambda$, then the canonical map

$$\operatorname*{colim}_\lambda \mathrm{Hom}_{G-\mathrm{alg}}(\overline{R}, R \otimes_A B_\lambda) \to \mathrm{Hom}_{G-\mathrm{alg}}(\overline{R}, R \otimes_A B)$$

is bijective. Since D and $\overline{R}$ are finitely generated k-algebras, any element of $\mathrm{Hom}_{G-\mathrm{alg}}(\overline{R}, R \otimes_A B_\lambda)$ whose image in $\mathrm{Hom}_{G-\mathrm{alg}}(\overline{R}, R \otimes_A B)$ lies in $F(B)$ has image in $\mathrm{Hom}_{G-\mathrm{alg}}(\overline{R}, R \otimes_A B_{\lambda'})$ which lies in $F(B_{\lambda'})$ for $\lambda' \geq \lambda$ sufficiently large. Thus F preserve filtered colimits.

The restriction $A \to \overline{R}$ of p to A factors through the augmentation $A \to k$. Thus the composite of the augmentation $\widehat{A} \to k$ with the embedding $k \to \overline{R}$ defines a homomorphism of G-algebras

$$\widehat{p} : \widehat{R} = R \otimes_A \widehat{A} \to \overline{R},$$

which factors as $p \otimes_A \widehat{A}$ followed by an isomorphism. The canonical homomorphism from R to $\widehat{R}$ then embeds D as a G-subalgebra of $\widehat{R}$ for which the restriction of $\widehat{p}$ to D is injective. Applying $(-)^G$ to the projection of $R \otimes_k \widehat{A}$ onto $\widehat{R}$ shows that $\widehat{R}^G$ is the image of $\widehat{A}$ in $\widehat{R}$. Since $\widehat{p}$ is surjective, we may thus apply (2) with $\widehat{R}$ for R and $\widehat{p}$ for p to obtain a G-subalgebra D' of $\widehat{R}$ such that the restriction of $\widehat{p}$ to D' is an isomorphism. The composite of the inverse of this isomorphism with the embedding of D' is then an element of $F(\widehat{A})$. Thus $F(\widehat{A})$ is non-empty.

(4) Write R_0 for the G-subalgebra of R generated by D and a lifting to R of a finite set of generators of $\overline{R}$. Then R_0 is a finitely generated G-algebra, so that $(R_0)^G$ is a finitely generated k-algebra. Since A is henselian, the embedding of $(R_0)^G$ into A extends to a homomorphism to A from the henselisation of $(R_0)^G$ at the kernel of $(R_0)^G \to A \to k$. Its image is a k-subalgebra A_0 of A containing $(R_0)^G$ which is the henselisation at a k-point of a k-scheme of finite type. If R_1 is the G-subalgebra of R generated by A_0 and R_0, then the restriction p_1 of p to R_1 is surjective, R_1 contains D, and R_1 is a finitely generated A_0-algebra. Applying $(-)^G$ to the canonical G-homomorphism from $R_0 \otimes_k A_0$ to R_1 shows further that $(R_1)^G = A_0$, Thus by (3) with R_1 and p_1 for R and p, there is a G-subalgebra $D' \supset D$ of $R_1 \subset R$ such that the restriction of p to D' is an isomorphism.

(5) Step (3) of the proof of [O'S11, Lemma 4.4.4] applies with "so that by (2) with $G/H_0 \dots$" on the last line of [O'S11, p. 45] replaced by by "so that by (4) with $G/H_0 \dots$". $\square$

We use the terminology of [O'S11] for tensor categories. Thus a *k-pretensor category* is a k-linear category with a bilinear tensor product together with unit, associativity and commutativity constraints. A *k-tensor category* is a pseudoabelian k-pretensor category, i.e. finite direct sums exist, and every idempotent endomorphism has an image. A k-pretensor category is said to be *rigid* if each of its objects is dualisable. In a rigid k-pretensor category $\mathcal{C}$, we may identify tensor ideals with subfunctors of $\mathcal{C}(\mathbf{1}, -)$, by assigning to $\mathcal{J}$ the subfunctor $\mathcal{J}(\mathbf{1}, -)$.

Let $\mathcal{C}$ be a rigid k-pretensor category for which $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ a local k-algebra. Then $\mathcal{C}$ has a unique maximal tensor ideal, which we write $\mathrm{Rad}(\mathcal{C})$. The elements of $\mathcal{C}(\mathbf{1}, M)$ lying in $\mathrm{Rad}(\mathcal{C})$ are those with no left inverse. We have $\mathrm{Rad}(\mathcal{C}) = 0$ if and only if $\mathcal{C}$ has no tensor ideals other than 0 and $\mathcal{C}$, and $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is then a field. In particular if $\mathrm{Rad}(\mathcal{C}) = 0$ then every k-tensor functor $\mathcal{C} \to \mathcal{C}'$ with $\mathcal{C}'$ non-zero is faithful. If $\mathcal{C}$ is semisimple abelian, then $\mathrm{Rad}(\mathcal{C}) = 0$, because $\mathbf{1}$ is indecomposable in $\mathcal{C}$ and hence every non-zero element of $\mathcal{C}(\mathbf{1}, M)$ has a left inverse. In general, we write

$$\overline{\mathcal{C}} = \mathcal{C}/\operatorname{Rad}(\mathcal{C}).$$

Then $\mathrm{Rad}(\overline{\mathcal{C}}) = 0$.

An object M of a k-tensor category is called *positive* if it is dualiasable and some exterior power (defined as the image of the antisymmetrising idempotent) of M is 0. The Cayley–Hamilton holds for such objects: if M is dualisable with $(n+1)$th power 0, then any endomorphism f of M is annulled by a monic polynomial of degree n, with coefficients the traces of exterior powers of f [O'S11, p.39]. Any vector bundle of bounded rank over a a k-scheme X is positive. It follows that if H is a pregroupoid over X with $H^0_H(X, \mathcal{O}_X)$ local, then any object M in $\mathrm{Mod}_H(X)$ is positive, because its rank $\mathrm{tr}(1_M)$ is constant.

Lemma 14.4. *Let $\mathcal{C}$ be a k-tensor category in which every object is positive. Suppose that $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is a local k-algebra. Then the projection from $\mathcal{C}$ onto $\overline{\mathcal{C}}$ reflects isomorphisms.*

Proof. It is enough to show that $1 + n$ is invertible for every $n : M \to M$ in $\mathcal{C}$ in the kernel of the projection. Write $\mathfrak{o}$ for $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ and $\mathfrak{o}'$ for the commutative $\mathfrak{o}$-subalgebra of $\mathrm{End}_{\mathcal{C}}(M)$ generated by n. By the Cayley–Hamilton Theorem, n is annulled by a polynomial with leading coefficient 1 and other coefficients in the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. Thus $\mathfrak{o}'$ is finite over $\mathfrak{o}$, and n is nilpotent in $\mathfrak{o}'/\mathfrak{m}\mathfrak{o}'$. It follows that $1 + n$ is invertible in $\mathfrak{o}'/\mathfrak{m}\mathfrak{o}'$, and hence in $\mathfrak{o}'$. $\square$

Lemma 14.5. *Let $\mathcal{C}$ be an essentially small k-tensor category in which every object is positive. Suppose that $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is a henselian local k-algebra with residue field k. Then $\overline{\mathcal{C}}$ is semisimple Tannakian.*

Proof. By the Cayley–Hamilton Theorem, any endomorphism of $\mathcal{C}$ is finite over $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$, and hence contained in a k-subalgebra of the endomorphism k-algebra which is finite product of local k-algebras. Thus idempotent endomorphisms can be lifted from $\overline{\mathcal{C}}$ to $\mathcal{C}$, so that $\overline{\mathcal{C}}$ is pseudo-abelian. Hence $\overline{\mathcal{C}}$ is a k-tensor category in which every object is positive. Since $\mathrm{End}_{\overline{\mathcal{C}}}(\mathbf{1}) = k$ and $\mathrm{Rad}(\overline{\mathcal{C}}) = 0$, it follows that $\overline{\mathcal{C}}$ is semisimple abelian, and hence semisimple Tannakian. $\square$

The following lemma is a modified form of [O'S11, Lemma 4.4.5]. It was there assumed that $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$, but there was a less stict condition on the objects than positivity. The first two paragraphs of the proof of [O'S11, Lemma 4.4.5] concern the reduction to the positive case, and are irrelevant here. The remaining three paragraphs may be applied unmodified to prove Lemma 14.6.

Lemma 14.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be essentially small k-tensor categories in which every object is positive. Suppose that $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is a henselian local k-algebra with residue field k. Then any lifting $\mathcal{D} \to \mathcal{C}$ along the projection $Q : \mathcal{C} \to \overline{\mathcal{C}}$ of a faithful k-tensor functor $\mathcal{D} \to \overline{\mathcal{C}}$ factors up to tensor isomorphism through some right inverse to Q.*

The following is the splitting theorem which will be used, together with some of the well-known properties of Tannakian categories and fibre functors, to prove the main theorem of the next section.

Theorem 14.7. *Let $\mathcal{C}$ be an essentially small k-tensor category with every object positive such that $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is a henselian local k-algebra with residue field k. Then the projection $Q : \mathcal{C} \to \overline{\mathcal{C}}$ has a right inverse, unique up to tensor isomorphism. If V is such a right inverse, then any k-tensor functor functor T from an essentially small k-tensor category to $\mathcal{C}$ with QT faithful factors up to tensor isomorphism through V, and such a factorisation is unique up to tensor isomorphism.*

Proof. The existence of a right inverse to Q follows by applying Lemma 14.6 with $\mathcal{D} = \mathrm{Mod}(k)$. Let V_1 and V_2 be two such right inverses. Then, as has been seen above, there is a k-tensor category $\mathcal{E}$ and a k-tensor functor $U : \mathcal{E} \to \mathcal{C}$ with QU faithful such that both V_1 and V_2 factor through U. Every object of $\mathcal{E}$ is positive, so that by Lemma 14.6 there is a right inverse W to Q through which U, and hence each of V_1 and V_2, factors up to tensor isomorphism. Composing with Q now shows that V_1 and V_2 are tensor isomorphic to W, and hence to each other.

If $\mathcal{D}$ is a k-tensor category and $T : \mathcal{D} \to \mathcal{C}$ is a k-tensor functor with QT faithful, then every object of $\mathcal{D}$ is positive. Since a right inverse to Q is unique up to tensor isomorphism, the existence of the required factorisation of T for $\mathcal{D}$ essentially small thus follows from Lemma 14.6. Its uniqueness up to tensor isomorphism follows by composing with Q. □

The basic facts from the theory of Tannakian categories that will be required are contained in Lemma 14.8 to 14.11 below. We write

$$h^* : \mathrm{Mod}_X(H) \to \mathrm{Mod}_X(H')$$

for the k-tensor functor induced by the morphism $h : H' \to H$ of groupoids over X, and

$$\omega_H : \mathrm{Mod}_H(X) \to \mathrm{Mod}(X).$$

for the forgetful functor. We have $\omega_H = \omega_{H'} h^*$ for any $h : H' \to H$. If K is a groupoid over H, we also write

$$\omega_{K/H} : \mathrm{Mod}_K(X) \to \mathrm{Mod}_H(X),$$

for f^*, where $f : H \to K$ is the structural morphism of K.

Lemma 14.8. *Let H be a pregroupoid and K a transitive affine groupoid over X. Then for every k-tensor functor $F : \mathrm{Mod}_K(X) \to \mathrm{Mod}_H(X)$ and tensor isomorphism $\varphi : \omega_H F \xrightarrow{\sim} \omega_K$ there is a morphism $f : H \to K$ of pregroupoids over X and a tensor isomorphism $\theta : F \xrightarrow{\sim} f^*$ such that $\omega_H \theta = \varphi$, and the pair (f, θ) is unique.*

Proof. There is a unique k-tensor functor F' from $\mathrm{Mod}_K(X)$ to $\mathrm{Mod}_H(X)$ such that the componenents of φ are at the same time the components of a tensor isomorphism $\theta : F \xrightarrow{\sim} F'$. Then $\omega_H\theta = \varphi$, so that $\omega_K = \omega_H F'$. It is thus enough to show that there is a unique f with $F' = f^*$. We have $F' = f^*$ if and only if for each S-point (s_0, s_1) of $X \times_k X$ the square

$$\begin{array}{ccc} H_{[1]}(S)_{(s_0,s_1)} & \longrightarrow & K_{[1]}(S)_{(s_0,s_1)} \\ \downarrow & & \downarrow \\ \mathrm{Iso}^{\otimes}({s_1}^*\omega_H, {s_0}^*\omega_H) & \longrightarrow & \mathrm{Iso}^{\otimes}({s_1}^*\omega_K, {s_0}^*\omega_K) \end{array}$$

commutes, where the top arrow is that induced on fibres above (s_0, s_1) by f, the bottom arrow sends θ to $\theta F'$, and the vertical arrows are the canonical ones: to see the "if" take $S = H_{[1]}$ and evaluate at the identity of $H_{[1]}$. Since the right arrow is an isomorphism by [Del90, 1.12(iii)], the existence and uniqueness of f follow. □

Lemma 14.9. *Let f_1 and f_2 be morphisms from a pregroupoid H to a transitive affine groupoid K over X. Then any tensor isomorphism $f_1^* \xrightarrow{\sim} f_2^*$ is induced by a cross section α of K^{diag} with $f_2 = \mathrm{int}(\alpha) \circ f_1$, and such an α is unique.*

Proof. Let $\theta : {f_1}^* \xrightarrow{\sim} {f_2}^*$ be a tensor isomorphism. By [Del90, 1.12(iii)], there is a unique cross section α of K^{diag} such that the tensor automorphism of ω_K induced by α is $\omega_H\theta$. If θ' is the tensor isomorphism from ${f_1}^*$ to $(\mathrm{int}(\alpha) \circ f_1)^*$ induced by α, we have $\omega_H\theta' = \omega_H\theta$. Lemma 14.8 with $F = {f_1}^*$ then shows that $f_2 = \mathrm{int}(\alpha) \circ f_1$. □

Lemma 14.10. *Let H be a pregroupoid over X and $T : \mathcal{D} \to \mathrm{Mod}_H(X)$ be a k-tensor functor with $\mathcal{D}$ Tannakian for which $\omega_H T$ is a fibre functor of $\mathcal{D}$. Then there is a transitive affine groupoid K over H such that T is the composite of a k-tensor equivalence from $\mathcal{D}$ to $\mathrm{Mod}_K(X)$ with $\omega_{K/H}$.*

Proof. Set $K = \underline{\mathrm{Aut}}^{\otimes}(\omega_H T)$. The action of H on the images under $\omega_H T$ of the objects of $\mathcal{D}$ defines a morphism $H \to K$ of pregroupoids over X, and hence a structure of groupoid over H on K. Then $T = \omega_{K/H} V$ with V the canonical k-tensor functor from $\mathcal{D}$ to $\mathrm{Mod}_K(X)$ associated to $\omega_H T$. Since $\omega_H T$ is a fibre functor, V is an equivalence by [Del90, 1.12(ii)]. □

Lemma 14.11. *Let H be a pregroupoid over X, K and K' be transitive affine groupoids over H, and F be a k-tensor functor from $\mathrm{Mod}_K(X)$ to $\mathrm{Mod}_{K'}(X)$ with $\omega_{K'/H}F$ tensor isomorphic to $\omega_{K/H}$. Then F is tensor isomorphic to f^* for some morphism $f : K' \to K$ over H, and any two such morphisms are conjugate.*

Proof. Let $\varphi : \omega_{K'/H}F \xrightarrow{\sim} \omega_{K/H}$ be a tensor isomorphism. By Lemma 14.8, we have a tensor isomorphism $\theta : F \xrightarrow{\sim} f^*$ with $\omega_H\theta = \omega_H\varphi$ for some $f : K' \to K$ over X. We have $\omega_{K/H} = h^*$ and $\omega_{K'/H} = h'^*$ with h and h' the structural morphisms of K and K'. Thus $(f \circ h', h'^*\theta)$ coincides with (h, φ) by the uniqueness statement of Lemma 14.8, so that f is a morphism over H. The uniqueness up to conjugacy follows from Lemma 14.9. □

It is convenient tp prove here following lemma, which will also be required later. For its proof we need the fact that if K is a reductive groupoid over X and $K' = K \times_X X'$ for an affine K-scheme X', then any representation $\mathcal{V}'$ of K' is projective

in $\mathrm{MOD}_{K'}(X')$. To see this, reduce using the fact that $\mathcal{V}'$ is dualisable to the case where $\mathcal{V}' = \mathcal{O}_{X'}$, and note that

$$\mathrm{Hom}_{K'}(\mathcal{O}_{X'}, -) = \mathrm{Hom}_{K,\mathcal{R}}(\mathcal{R}, -) = \mathrm{Hom}_K(\mathcal{O}_X, -)$$

by the equivalence (8.6), where $X' = \mathrm{Spec}(\mathcal{R})$.

Lemma 14.12. *Let K be a reductive groupoid over X and X' be an affine K-scheme. Then every representation of $K \times_X X'$ is a direct summand of the pullback along $X' \to X$ of a representation of K.*

Proof. Write $X' = \mathrm{Spec}(\mathcal{R})$ for a K-algebra $\mathcal{R}$. By the equivalence (8.8), it is enough to show that every representation $\mathcal{V}$ of $(K, \mathcal{R})$ is a direct summand of $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}_0$ for some representation $\mathcal{V}_0$ of K. We have an epimorphism of $(K, \mathcal{R})$-modules

$$\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V} \to \mathcal{V}.$$

Writing the K-module $\mathcal{V}$ as the filtered colimit of its subrepresentations $\mathcal{V}_0$, we obtain an epimorphism, and hence by the above remark a retraction, from $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}_0$ to $\mathcal{V}$ for some $\mathcal{V}_0$. □

15. Universal and minimal reductive groupoids

In this section k is a field of characteristic 0, X is a non-empty k-scheme, and H is a pregroupoid over X.

In this section we prove that a universal reductive groupoid over H exists provided that $H^0_H(X, \mathcal{O}_X)$ is a henselian local k-algebra with residue field k. As well as the classification theorems of Sections 19 and 20, this result has many consequences which will developed in this and the next three sections. A particularly import application is to those reductive groupoids over H which do not properly contain a reductive subgroupoid over H, and their corresponding principal bundles.

Definition 15.1. Let K be a reductive groupoid over H. We say that K is *universally reductive* if for every reductive groupoid K' over H there exists a morphism from K to K' over H, and if such a morphism is unique up to conjugacy. We say that K is *minimally reductive* if it does not strictly contain a reductive subgroupoid over H.

Let K be a universally reductive groupoid over H. If K is a reductive groupoid over H, there is thus a morphism f, unique up to conjugacy from K to K'. Then K' is minimally reductive if and only if f is surjective.

Proposition 15.2. *Let K be a universal reductive groupoid over H. Then any two morphisms over H from K to a transitive affine groupoid over H are conjugate.*

Proof. Let K' be a transitive affine groupoid over H such that a morphism from K to K' over H exists. Then K' has a Levi subgroupoid K'' over H, by Theorem 13.4. By Proposition 12.3, any morphism from K to K' over H is conjugate to one which factors through K''. The result thus follows by the universal property of H. □

The following result gives sufficient conditions under which Krull–Schmidt holds for representations of H, i.e. under which the commutative monoid of isomorphism classes of representations of H under direct sum is free. Note that a pseudo-abelian category has the Krull-Schmidt property if and only if each endomorphism ring R satisfies the following condition: the identity of R has an decomposition into

pairwise orthogonal irreducible idempotents, and any two such decompositions are conjugate in R. The condition on R is satisfied in particular if R is artinian.

Proposition 15.3. *Suppose that $H^0_H(X,\mathcal{O}_X)$ is a henselian local k-algebra. Then any representation of H is a finite direct sum of indecomposable representations of H, and such a decomposition is unique up to isomorphism.*

Proof. Replacing k by a sufficiently large extension contained in $H^0_H(X,\mathcal{O}_X)$, we may suppose that $H^0_H(X,\mathcal{O}_X)$ has residue field k. Then by Lemma 14.5, $\overline{\mathrm{Mod}_H(X)}$ is a semisimple Tannakian k-tensor category, and in particular pseudo-abelian with finite-dimensional hom k-spaces. Thus $\overline{\mathrm{Mod}_H(X)}$ has the Krull–Schmidt property. Since by Lemma 14.4 the projection from $\mathrm{Mod}_H(X)$ to $\overline{\mathrm{Mod}_H(X)}$ induces a bijection on isomorphism classes of objects, it follows $\mathrm{Mod}_H(X)$ has the Krull–Schmidt property. □

Theorem 15.4. *Suppose that $H^0_H(X,\mathcal{O}_X)$ is a henselian local k-algebra with residue field k.*

(i) *A universal reductive groupoid over H exists.*
(ii) *A transitive affine groupoid K over H is universally (resp. minimally) reductive over H if and only if $\omega_{K/H}$ composed with the projection from $\mathrm{Mod}_H(X)$ to $\overline{\mathrm{Mod}_H(X)}$ is an equivalence of categories (resp. fully faithful).*

Proof. Consider the category $\mathcal{T}_H$ whose objects are pairs $(\mathcal{D},T)$ with $\mathcal{D}$ a semisimple Tannakian k-tensor category and T a k-tensor functor from $\mathcal{D}$ to $\mathrm{Mod}_H(X)$, where a morphism from $(\mathcal{D},T)$ to $(\mathcal{D}',T')$ is a tensor isomorphism class of functors $F : \mathcal{D} \to \mathcal{D}'$ with $T'F$ tensor isomorphic to T. If $\mathcal{R}_H$ is the category of reductive groupoids over H up to conjugacy, we have a functor $\mathcal{R}_H \to \mathcal{T}_H{}^{\mathrm{opp}}$ which sends K to $(\mathrm{Mod}_K(X),\omega_{K/H})$ and and the class of $f : K' \to K$ to the class of f^*. This functor is essentially surjective by Lemma 14.10 and fully faithful by Lemma 14.11. Thus if K is a reductive groupoid over H, then K is universally reductive $\iff$ K is initial in $\mathcal{R}_H$ $\iff$ $(\mathrm{Mod}_K(X),\omega_{K/H})$ is final in $\mathcal{T}_H$. By Theorem 14.7, $\mathcal{T}_H$ has a final object, and if Q is the projection, then $(\mathcal{D},T)$ is final $\iff$ QT is an equivalence. Hence (i) and the criterion of (ii) for universality.

To prove the criterion of (ii) for a minimality we may suppose that K is reductive. By (i), there is a morphism $f : K' \to K$ over H with K' universally reductive over H. Then K minimally reductive over H $\iff$ f is surjective $\iff$ f^* is fully faithful (K' is reductive) $\iff$ $Q\omega_{K'/H}f^* = Q\omega_{K/H}$ is fully faithful. □

Let $T : \mathcal{D} \to \mathcal{D}'$ be a faithful k-tensor functor between Tannakian k-tensor categories. Suppose that $\mathcal{D}'$ is semisimple. Then T is fully faithful if and only if it sends indecomposable objects of $\mathcal{D}$ not isomorphic to $\mathbf{1}$ to indecomposable objects of $\mathcal{D}'$ not isomorphic to $\mathbf{1}$. Indeed the "only if" is clear and the "if" follows because T is fully faithful provided that it induces a bijection

$$\mathrm{Hom}_{\mathcal{D}}(\mathbf{1},M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}'}(\mathbf{1},T(M))$$

for M indecomposable in $\mathcal{D}$.

Corollary 15.5. *Let H be as in Theorem 15.4 and K be a reductive groupoid over H. Then the following conditions are equivalent.*

(a) *K is universally reductive over H.*
(b) *$\omega_{K/H}$ induces a bijection on isomorphism classes of representations.*

(c) *K is minimally reductive over H, and every representation of H is a direct summand of a representation of K.*

Proof. The projection Q from $\mathrm{Mod}_H(X)$ to $\overline{\mathrm{Mod}_H(X)}$ induces a bijection on isomorphism classes of objects. Thus the implication (a) $\implies$ (b) follows from Theorem 15.4(ii). If (b) holds then $Q\omega_{K/H}$ sends indecomposables not isomorphic to $\mathbf{1}$ to indecomposables not isomorphic to $\mathbf{1}$, and hence is fully faithful by the criterion above. The implication (b) $\implies$ (c) thus follows from Theorem 15.4(ii). The implication (c) $\implies$ (a) also follows from Theorem 15.4(ii). $\square$

Recall that ${}^{\mathrm{rad}}H^0_H(X,\mathcal{V})$ denotes the kernel on the right of the pairing (9.2).

Corollary 15.6. *Let H be as in Theorem* 15.4 *and K be a reductive groupoid over H. Then the following conditions are equivalent.*

(a) *K is minimally reductive over H.*
(b) *$H^0_H(X,\mathcal{V}) = {}^{\mathrm{rad}}H^0_H(X,\mathcal{V}) \oplus H^0_K(X,\mathcal{V})$ for every representation $\mathcal{V}$ of K.*
(c) *Every non-trivial indecomposable representation of K is a non-trivial indecomposable representation of H.*

Proof. Write Q for the projection from $\mathrm{Mod}_H(X)$ to $\overline{\mathrm{Mod}_H(X)}$. Then each of (a), (b), and (c) is equivalent to the condition that $Q\omega_{K/H}$ be fully faithful: in the case of (a) by Theorem 15.4(ii), in the case of (b) because (b) is equivalent to requiring that $Q\omega_{K/H}$ induce an isomorphism on the hom-spaces $\mathrm{Hom}_K(\mathcal{O}_X,\mathcal{V})$, and in the case of (c) by the criterion preceding Corollary 15.5. $\square$

Remark 15.7. Corollary 15.6 can be used to decompose the tensor powers of a vector bundle into its indecomposable summands by reducing to a question about representations of reductive groups. The following is a simple example, which will be useful in Section 20. Let H be as in Theorem 15.4 and suppose that k is algebraically closed and that every finite étale H-scheme of rank 2 is trivial. Then every symmetric power of an indecomposable representation $\mathcal{V}$ of H of rank 2 is indecomposable. To see this, let K be a minimal reductive subgroupoid over H of $\underline{\mathrm{Iso}}_{\mathcal{O}_X}(\mathcal{V})$. By Lemma 10.6(i) and Corollary 15.6, we may replace H by K and hence suppose that H is a reductive groupoid. Then by Lemma 7.4, $H = \underline{\mathrm{Iso}}_G(P)$ for a reductive k-group G and principal G-bundle P. Pulling back onto P, we reduce first to the case where P is trivial and then to the case where $X = \mathrm{Spec}(k)$ and $H = G$ is a reductive k-group. The image $\overline{H}$ of H in $GL(\mathcal{V})$ is non-commutative, because otherwise $\mathcal{V}$ would be decomposable. Also $\overline{H}$ has no finite quotient of order 2, because such a quotient would define a non-trivial finite étale H-scheme of rank 2. Thus $\overline{H}$ is either $GL(\mathcal{V})$ or $SL(\mathcal{V})$, and the result follows.

Corollary 15.8. *Let H be as in Theorem* 15.4 *and K be a minimal reductive groupoid over H.*

(i) *Any two morphisms over H from K to a transitive affine groupoid over H are conjugate.*
(ii) *Any two representations of K which are H-isomorphic are K-isomorphic.*

Proof. A universal reductive groupoid K' over H exists, by Theorem 15.4(i). There is a surjective morphism $h : K' \to K$ over H. Composing with h and using Proposition 15.2 gives (i), while composing with h and using Corollary 15.5(b) gives (ii). $\square$

Corollary 15.9. *Let H be as in Theorem* 15.4 *and K be a transitive affine groupoid over H. Then any two minimal reductive subgroupoids of K over H are conjugate by an element of $H^0_H(X, K^{\mathrm{diag}})$.*

Proof. By Theorem 15.4(i), a universal reductive groupoid K' over H exists. Any minimal reductive subgroupoid of K over H is the image of a morphism $K' \to K$ over H. By Proposition 15.2, any two morphisms $K' \to K$ over H are conjugate by a cross section of the diagonal of K, which necessarily lies in $H^0_H(X, K^{\mathrm{diag}})$. □

Remark 15.10. That some condition on $H^0_H(X, \mathcal{O}_X)$ is necessary in order that the above Corollaries should hold can be seen as follows. Suppose that k is algebraically closed, and write $G = GL_2$ and K for the normaliser of a maximal torus in SL_2. We regard K as a k-subgroup of G by the embedding of SL_2, and take $X = G/K$. Then G is a principal K-bundle over X whose push forward along the embedding $K \to G$ is a trivial G-bundle. The vector bundle over X defined by G and the irreducible 2-dimensional representation $K \to G$ of K is thus trivial. On the other hand, the structure group of G over X cannot be reduced to a proper k-subgroup K' of K. Indeed such a reduction would correspond to to a K-morphism $G \to K' \backslash K$. Such a K-morphism is necessarily surjective, so that since G is connected $K' \backslash K$ is necessarily isomorphic as a k-scheme to $\mathbf{G}_m$. Thus $G \to K' \backslash K$ is constant along each coset of SL_2, which is impossible because K is contained in SL_2.

Let H be a pregroupoid over X. For every H-module $\mathcal{V}$ we have a pairing

$$\mathrm{Hom}_H(\mathcal{V}, \mathcal{O}_X) \otimes_k H^0_H(X, \mathcal{V}) \to H^0_H(X, \mathcal{O}_X)$$

defined by composition. Its image is an ideal

$$I_H(\mathcal{V}) \subset H^0_H(X, \mathcal{O}_X).$$

We have

$$I_H(\mathcal{V}) + I_H(\mathcal{W}) \subset I_H(\mathcal{V} \oplus \mathcal{W})$$

for every $\mathcal{V}$ and $\mathcal{W}$.

Let K be a reductive groupoid over H. We write

$$I_H(K) := \bigcup_{\mathcal{V} \in \mathrm{Mod}_K(X),\ H^0_K(X, \mathcal{V}) = 0} I_H(\mathcal{V}) \ \subset H^0_H(X, \mathcal{O}_X)$$

for the union of the $I_H(\mathcal{V})$ where $\mathcal{V}$ runs over those representations of K which have no direct summand isomorphic to the identity. Then $I_H(K)$ is an ideal of $H^0_H(X, \mathcal{O}_X)$.

We note that if M and N are objects in a pseudo-abelian category, then the composition homomorphism

$$\mathrm{Hom}(M, N) \otimes \mathrm{Hom}(N, M) \to \mathrm{End}(N) \tag{15.1}$$

is surjective if and only if N is a direct summand of M^n for some n.

Proposition 15.11. *Let K be a reductive groupoid over H. Then the following two conditions are equivalent.*

(a) $I_H(K) \neq H^0_H(X, \mathcal{O}_X)$.
(b) *For every representation of K with no non-zero trivial direct summand, the underlying representation of H has no non-zero trivial direct summand.*

Proof. Taking for M in (15.1) the representation $\mathcal{V}$ of H and for N the trivial representation $\mathcal{O}_X$ of H shows that $I_H(\mathcal{V}) = H^0_H(X, \mathcal{O}_X)$ if and only if $\mathcal{V}^n$ has the direct summand $\mathcal{O}_X$ for some n. The result follows. □

Let k' be an extension of k contained in $H^0(X, \mathcal{O}_X)$. Then we may regard X as a k'-scheme. A pregroupoid H over X is then a pregroupoid in k'-schemes if and only if k' lies in $H^0_H(X, \mathcal{O}_X)$. Suppose that this condition holds, and let K be a reductive groupoid over H. The restriction K' of K to the subscheme $X \times_{k'} X$ of $X \times_k X$ is a reductive groupoid in k'-schemes over H. We have

$$I_H(K) = I_H(K'). \tag{15.2}$$

Indeed K' coincides, as a groupoid in k'-schemes over X, with the pullback of $K_{k'}$ along the canonical morphism of k'-schemes $X \to X_{k'}$. Thus $X \to X_{k'}$ defines an equivalence from $\mathrm{Mod}_{K_{k'}}(X_{k'})$ to $\mathrm{Mod}_{K'}(X)$, so that any representation of K' with no direct summand isomorphic to the identity is a direct summand of a representation of K with no direct summand isomorphic to the identity.

Let k' be an extension of k. Then Lemma 9.4 applied to the system $(\mathcal{O}_X \otimes_k V)$ where V runs over the finite-dimensional k-vector subspaces of k' shows that the embedding of $H^0_H(X, \mathcal{O}_X)$ induces an isomorphism

$$H^0_H(X, \mathcal{O}_X)_{k'} \xrightarrow{\sim} H^0_{H_{k'}}(X_{k'}, \mathcal{O}_{X_{k'}}) \tag{15.3}$$

provided that either k' is finite over k or X is quasi-compact.

Proposition 15.12. *Let K be a reductive groupoid over H and k' be an extension of k. Suppose that ether k' is finite over k or that X is quasi-compact. Then $I_{H_{k'}}(K_{k'})$ is the image of $I_H(K)_{k'}$ under* (15.3).

Proof. By Lemma 14.12 with $X' = X_{k'}$, every representation of $K_{k'}$ is a direct summand of $\mathcal{V}_{k'}$ for some representation $\mathcal{V}$ of K. □

Corollary 15.13. *Let K be a reductive groupoid over H. If $I_H(K) \neq H^0_H(X, \mathcal{O}_X)$, then K is minimally reductive over H. The converse holds when H is as in Theorem* 15.4.

Proof. Suppose that K contains a reductive subgroupoid $K' \neq K$ over H. Then the functor from $\mathrm{Mod}_K(X)$ to $\mathrm{Mod}_{K'}(X)$ defined by the embedding is not fully faithful. Thus by the criterion preceding Corollary 15.5 there is a representation $\mathcal{V}$ of K with no non-zero trivial direct summand such that the underlying representation of K', and hence the underlying representation of H, has the trivial direct summand $\mathcal{O}_X$. This implies $I_H(K) = H^0_H(X, \mathcal{O}_X)$, by Proposition 15.11.

If H is as in Theorem 15.4 and K is minimally reductive over H, then by Corollary 15.6(b) $I_H(\mathcal{V}) \neq H^0_H(X, \mathcal{O}_X)$ for $\mathcal{V}$ a representation of K with no non-zero direct summand. Thus $I_H(K) \neq H^0_H(X, \mathcal{O}_X)$. □

Remark 15.14. Suppose that X is affine. Then $I_X(\mathcal{V}) = H^0(X, \mathcal{O}_X)$ for every $\mathcal{V} \neq 0$ in $\mathrm{Mod}(X)$. Thus

$$I_X(K) = H^0(X, \mathcal{O}_X)$$

for every reductive groupoid K over X other than the constant groupoid 1. Suppose that a non-constant reductive groupoid over X exists, as is the case for example when there is a $\mathcal{V}$ in $\mathrm{Mod}(X)$ which is not free. Then a minimal reductive groupoid over X exists other than the constant groupoid 1. This shows that some condition on H is necessary for the final statement of Corollary 15.13.

Remark 15.15. Suppose that $H^0_H(X, \mathcal{O}_X)$ is a henselian local k-algebra. Denote by k' its residue field. Then there exists an embedding over k of k' into $H^0_H(X, \mathcal{O}_X)$ which is right inverse to the projection. Choose such an embedding. Then we may regard H as a pregroupoid in k'-schemes, and if K is a reductive groupoid over H and K' is the restriction of K to the subscheme $X \times_{k'} X$ of $X \times_k X$ we have the equality (15.2). Thus by Corollary 15.13 the conditions

- $I_H(K) \neq H^0_H(X, \mathcal{O}_X)$
- K' is minimally reductive over H

are equivalent. These conditions are in general strictly stronger than the one that K be minimally reductive over H. The following are two examples of this, with $X = \mathrm{Spec}(k')$ and $H = X$. Suppose first that k is algebraically closed but that k' is not. Then a non-trivial principal bundle with reductive structure group over X exists, such as the spectrum of a non-trivial Galois extension of k' with structure group the Galois group. A non-constant reductive groupoid over X, and hence a minimal reductive groupoid K over X other than the constant groupoid 1, thus exists. On the other hand reductive groupoids over the k'-scheme X are just reductive k'-groups, so that $K' \neq 1$ is not minimally reductive. For the second example suppose that k' is a quadratic extension of k. Then the non-zero element of

$$H^2(\mathrm{Gal}(k'/k), \mathbf{Z}/2) = \mathbf{Z}/2$$

defines a non-neutral reductive groupoid K over X with K^{diag} the discrete k'-group of order 2, Then K is minimally reductive, but K', and even $K_{k'}$, is not.

We may regard H as a groupoid in S-schemes, where

$$S = \mathrm{Spec}(A) = \mathrm{Spec}(H^0_H(X, \mathcal{O}_X)).$$

If S' is a scheme over S and $X' = X \times_S S'$ is the constant H-scheme defined by S', then we have a canonical isomorphism

$$H' = H \times_S S' \xrightarrow{\sim} H \times_X X'$$

of pregroupoids over X'. Suppose that $S' = \mathrm{Spec}(A')$ is affine and flat over S, and that X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact. Then for any representation $\mathcal{V}$ of H with pullback $\mathcal{V}'$ onto X', the canonical homomorphism from $H^0_H(X, \mathcal{V})$ to $H^0_{H'}(X', \mathcal{V}')$ extends to an isomorphism

$$H^0_H(X, \mathcal{V}) \otimes_A A' \xrightarrow{\sim} H^0_{H'}(X', \mathcal{V}').$$

This follows from the fact that $H^0_H(X, \mathcal{V})$ for example is the equaliser of two morphisms from $H^0(X, \mathcal{V})$ to $H^0(H_{[1]}, d_1{}^*\mathcal{V})$, while base change along $S' \to S$ is an isomorphism for X and $\mathcal{V}$ and injective for $H_{[1]}$ and $d_1{}^*\mathcal{V}$. In particular we have a canonical isomorphism

$$A' \xrightarrow{\sim} H^0_{H'}(X', \mathcal{O}_{X'}),$$

which induces for every reductive groupoid K over H an isomorphism

$$I_H(K) \otimes_A A' \xrightarrow{\sim} I_{H'}(K \times_{[X]} [X']).$$

This can be used as follows to describe the support of the subscheme of S defined by $I_H(K)$. Let s be a point of S. Write S_s for the spectrum of the henselisation of S at s and X_s for $X \times_S S_s$, and let k_s be a k-subalgebra of this henselisation

with projection onto the residue field an isomorphism. Then by the criterion of Remark 15.15, s lies in the support of $\mathrm{Spec}(A/I_H(K))$ if and only the groupoid

$$K \times_{[X]} [X_s]_{/k_s}$$

in k_s-schemes over $H \times_S S_s$ is minimally reductive.

Corollary 15.16. *Let H be as in Theorem* 15.4*, K be a reductive groupoid over H, and k' be an extension of k. Suppose that either k' is finite over k or that X is quasi-compact. Then $K_{k'}$ is minimally reductive over $H_{k'}$ if and only if K is minimally reductive over H.*

Proof. That $K_{k'}$ minimally reductive over $H_{k'}$ implies K minimally reductive over H is immediate, even without any condition on k' or X. The converse follows from Proposition 15.12 and Corollary 15.13. □

Remark 15.17. That some condition on H is needed in Corollary 15.16 can be seen by the example of Remark 15.15 with $H = X$ and X the spectrum of a quadratic extension of k. The following is an example with $H = X$ and X geometrically connected. Suppose that $X = \mathbf{A}^2$ is the affine plane, and that k is not algebraically closed. Then by [Rag89, Theorem B], there exists a reductive k-group G and a principal G-bundle P over X such that $P_{\bar{k}}$ is a trivial $G_{\bar{k}}$-bundle over $X_{\bar{k}}$, but P is not the pullback of a principal G-bundle over $\mathrm{Spec}(k)$. Replacing G by an appropriate inner form and P by a twist, we may suppose that P has a k-point. Then the groupoid $\underline{\mathrm{Iso}}_G(P)$ is not constant. It thus has a minimally reductive subgroupoid K which is not constant. On the other hand the groupoid $K_{\bar{k}}$ over $X_{\bar{k}}/\bar{k}$ is constant, because every principal bundle over $X_{\bar{k}}/\bar{k}$ is trivial [Rag78].

Note that the condition on H in Theorem 15.4 is not in general preserved by extension of scalars. However by (15.3) it is preserved by algebraic extension of scalars, provided that either the extension is finite or X is quasi-compact.

Corollary 15.18. *Let H be as in Theorem* 15.4*, K be a reductive groupoid over H, and k' be an* algebraic *extension of k. Suppose that either k' is finite over k or that X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact. Then $K_{k'}$ is universally reductive over $H_{k'}$ if and only if K is universally reductive over H.*

Proof. By (c) of Corollary 15.5 and Corollary 15.16, it suffices to apply Lemma 9.7 with $H' = K$ and $X' = X_{k'}$. □

16. Applications to principal bundles

In this section k is a field of characteristic 0*, $\overline{k}$ ia an algebraically closed extension of k, X is a non-empty k-scheme, and H is a pregroupoid over X.*

By considering groupoids of the form $\underline{\mathrm{Iso}}_G(P)$, we can derive from Theorem 15.4 and its corollaries results applicable to principal bundles. If we consider only reductive k-groups G, it is necessary for this either to suppose that k is algebraically closed, or to suppose that X has a k-point x and confine attention to bundles which have a k-point above x. For the general case, it is necessary to consider $\underline{\mathrm{Iso}}_F(P)$ for reductive groupoids F over $\overline{k}$.

We first prove some criteria for a reductive groupoid K over H to be universally reductive. To do this we consider the map

$$\mathrm{Hom}^{\mathrm{conj}}_{X/k}(K, \underline{\mathrm{Iso}}_G(P)) \to \mathrm{Hom}^{\mathrm{conj}}_{X/k}(H, \underline{\mathrm{Iso}}_G(P)) \tag{16.1}$$

induced by $H \to K$ for an affine k-group G, and the map

$$\mathrm{Hom}^{\mathrm{conj}}_{X/k}(K, \underline{\mathrm{Iso}}_F(P)) \to \mathrm{Hom}^{\mathrm{conj}}_{X/k}(H, \underline{\mathrm{Iso}}_F(P)) \tag{16.2}$$

induced by $H \to K$ for a transitive affine groupoid F over $\overline{k}$.

Lemma 16.1. *Let K be a reductive groupoid over H and x be a k-point of X. Then K is universally reductive over H if and only if the canonical map from $\widetilde{H}^1_K(X, x, G)$ to $\widetilde{H}^1_H(X, x, G)$ is bijective for every reductive k-group G.*

Proof. By (5.6), K is universally reductive over H if and only if (16.1) is bijective for every reductive k-group G and principal G-bundle P over X trivial above x. This is equivalent to the bijectivity of the canonical map, by Lemma 5.1 and the naturality of (5.2) applied to $H \to K$. □

Lemma 16.2. *A reductive groupoid K over H is universally reductive over H if and only if the canonical map from $H^1_K(X, G)$ to $H^1_H(X, G)$ is bijective for every reductive groupoid G over $\overline{k}$.*

Proof. By Lemma 7.4, K is universally reductive over H if and only if (16.2) is bijective for every reductive groupoid F over $\overline{k}$ and principal F-bundle P over X. This is equivalent to the bijectivity of the canonical map, by Lemma 6.10 and the naturality of (6.2) applied to $H \to K$. □

Lemma 16.3. *Let G be a reductive k-group and P be a principal (H, G)-bundle over X which has a k-point above the k-point x of X. Then $\underline{\mathrm{Iso}}_G(P)$ is universally reductive over H if and only if G and $[P]$ represent the functor $\widetilde{H}^1_H(X, x, -)$ on reductive k-groups up to conjugacy.*

Proof. By Lemma 5.3(ii), G and the class of P represent $\widetilde{H}^1_{\underline{\mathrm{Iso}}_G(P)}(X, x, -)$ on reductive k-groups up to conjugacy. The result thus follows from Lemma 16.1. □

Lemma 16.4. *Let F be a reductive groupoid over $\overline{k}$ and P be a principal (H, F)-bundle over X. Then $\underline{\mathrm{Iso}}_F(P)$ is universally reductive over H if and only if F and $[P]$ represent the functor $H^1_H(X, -)$ on reductive groupoids over $\overline{k}$ up to conjugacy.*

Proof. By Lemma 7.5(ii), F and the class of P represent $H^1_{\underline{\mathrm{Iso}}_F(P)}(X, -)$ on reductive groupoids over $\overline{k}/k$ up to conjugacy. The result thus follows from Lemma 16.2. □

Theorem 16.5. *Let H be as in Theorem* 15.4*.*

(i) *For each k-point x of X the functor $\widetilde{H}^1_H(X, x, -)$ on reductive k-groups up to conjugacy is representable.*
(ii) *The functor $H^1_H(X, -)$ on reductive groupoids over $\overline{k}$ up to conjugacy is representable.*

Proof. By Theorem 15.4(i) and Lemmas 16.1 and 16.2, we may suppose that H is a reductive groupoid. Then (i) follows from Lemma 5.3 and (ii) from Lemma 7.5. □

Lemma 16.6. *Let H be a pregroupoid over X, x be a k-point of X, G be a reductive k-group, and P be a principal (H, G)-bundle with a k-point above x. Then $\underline{\mathrm{Iso}}_G(P)$ is minimally reductive over H if and only if P has no principal (H, G')-subbundle with a k-point above x for any reductive k-subgroup $G' \neq G$ of G.*

Proof. The "only if" is immediate. The "if" follows from the equivalence (5.5). □

Lemma 16.7. *Let H be a pregroupoid over X, F be a reductive groupoid over $\overline{k}$, and P be a principal (H,F)-bundle. Then $\underline{\mathrm{Iso}}_F(P)$ is minimally reductive over H if and only if P has no principal (H,F')-subbundle for any reductive subgroupoid $F' \neq G$ over $\overline{k}$ of F.*

Proof. The "only if" is immediate. The "if" follows from Lemma 7.4. □

Lemma 16.8. *Let K be a transitive affine groupoid over H. Then the following conditions are equivalent.*

(a) *Any two morphisms over H from K to a transitive affine groupoid over H are conjugate.*
(b) *The canonical map $H^1_K(X,F) \to H^1_H(X,F)$ is injective for every transitive affine groupoid F over $\overline{k}$.*

Proof. By Lemma 7.4, (a) holds if and only if (16.2) is injective for every transitive affine groupoid F over $\overline{k}$ and principal F-bundle P over X. This is equivalent to (b) by Lemma 6.10 and the naturality of (6.2) applied to $H \to K$. □

Corollary 16.9. *Let H be as in Theorem* 15.4 *and K be a minimal reductive groupoid over H. Then the canonical map $H^1_K(X,F) \to H^1_H(X,F)$ is injective for every transitive affine groupoid F over $\overline{k}$.*

Proof. Immediate from Corollary 15.8(i) and Lemma 16.8. □

Corollary 16.10. *Let H be as in Theorem* 15.4*, F be a reductive groupoid over $\overline{k}$, and P be a principal (H,F)-bundle such that $\underline{\mathrm{Iso}}_F(P)$ is minimally reductive over H. Then the push forwards of P along two morphisms h_1 and h_2 from F to a transitive affine groupoid F' over $\overline{k}$ are (H,F')-isomorphic if and only if h_1 and h_2 are conjugate.*

Proof. The map that sends the conjugacy class of $h : F \to F'$ to the cohomology class of the push forward of P along h factors as

$$\mathrm{Hom}_k^{\mathrm{conj}}(F,F') \xrightarrow{\sim} H^1_{\underline{\mathrm{Iso}}_F(P)}(X,F') \to H^1_H(X,F'),$$

where the first map is defined by $[P]$ and is bijective by Lemma 7.5, and the second is the canonical map and is injective by Corollary 16.9. □

Remark 16.11. In general, the equivalent conditions of Lemma 16.8 do not imply that K is minimally reductive over H, even for H as in Theorem 15.4. Suppose for example that k is algebraically closed and $\overline{k} = k$. Then $K = \underline{\mathrm{Iso}}_G(P)$ for an affine k-group G by Lemma 7.4, and (b) of Lemma 16.8 holds if and only if the map

$$i(G,P) : \mathrm{Hom}_k^{\mathrm{conj}}(G,G') \to H^1_H(X,G')$$

defined by P is injective for every affine k-group G'. Let P_0 be a principal $(H,\mathbf{G}_m)$-bundle whose class in $H^1_H(X,\mathbf{G}_m)$ is not of finite order, and P be the push forward of P_0 along an embedding $e : \mathbf{G}_m \to SL_2$. Then $\underline{\mathrm{Iso}}_{\mathbf{G}_m}(P_0)$ is minimally reductive over H but $\underline{\mathrm{Iso}}_{SL_2}(P)$ is not. Now by a theorem of Malcev, e induces an injective map

$$\mathrm{Hom}_k^{\mathrm{conj}}(SL_2,G') \to \mathrm{Hom}_k^{\mathrm{conj}}(\mathbf{G}_m,G')$$

for every k-group G'. Thus $i(SL_2,P)$ is injective because $i(\mathbf{G}_m,P_0)$ is injective.

Let F be a transitive affine groupoid over an extension k' of k, and P be principal F-bundle over X. Given a representation V of F, we define the vector bundle

$$P \times^F V$$

over X associated to V by identifying the points (pg, v) and (p, gv) of $P \times V$ for every point g of F. Explicitly, $P \times^F V$ is obtained by faithfully flat descent along $P \to X$ of the constant vector bundle $P \times V$ on P according to the isomorphism

$$(p_0, p_1, v) \mapsto (p_0, p_1, gv)$$

of its pullback along $(p_0, p_1) \mapsto p_1$ to its pullback along $(p_0, p_1) \mapsto p_0$, where g is defined using (6.3) by $p_1 = p_0 g$. Formation of $P \times^F V$ is functorial in P and V, and is compatible with pullback of V and F. If $k' = k$, then $P \times^F V$ is the usual vector bundle associated to a representation of the k-group F.

Suppose that P is a principal (H, F)-bundle. Then $P \times^F V$ has a canonical structure of representation of H, with H acting through its action on P. We then have a functor

$$P \times^F - : \mathrm{Mod}_F(k') \to \mathrm{Mod}_H(X).$$

It sends the trivial representation k' of G to the trivial representation $\mathcal{O}_X$ of H, so that there is an embedding

$$V^F := H^0_F(k', V) \to H^0_H(X, P \times^F V) \tag{16.3}$$

defined by functoriality.

The canonical structure of principal $(\underline{\mathrm{Iso}}_F(P), F)$-bundle on P defines a canonical structure of representation of $\underline{\mathrm{Iso}}_F(P)$ on $P \times^F V$. If P' is the pullback of P along $X' \to X$, then the canonical isomorphism from the pullback of $P \times^F V$ onto X' to $P' \times^F V$ is an isomorphism of representations of $\underline{\mathrm{Iso}}_F(P')$ which is natural in V. Taking $X' = P$ so that P' is trivial, and using the fact that pullback functors for representations of transitive groupoids are equivalences, now shows that

$$P \times^F - : \mathrm{Mod}_F(k') \to \mathrm{Mod}_{\underline{\mathrm{Iso}}_F(P)}(X) \tag{16.4}$$

is an equivalence of categories. In particular when $H = \underline{\mathrm{Iso}}_F(P)$, (16.3) is an isomorphism.

Corollary 16.12. *Let H be as in Theorem* 15.4*, F be a reductive groupoid over $\overline{k}$, and P be a principal (H, F)-bundle. Then the following conditions are equivalent.*

(a) *$\underline{\mathrm{Iso}}_F(P)$ is universally reductive over H.*
(b) *$P \times^F -$ induces a bijection from isomorphism classes of representations of F to isomorphism classes of representations of H.*
(c) *$\underline{\mathrm{Iso}}_F(P)$ is minimally reductive over H, and every representation of H is a direct summand of $P \times^F V$ for some representation V of F.*

Proof. It is enough to show that with $K = \underline{\mathrm{Iso}}_F(P)$, each of (b) and (c) is equivalent to the corresponding condition of Corollary 15.5. This is clear because (16.4) is an equivalence. □

Corollary 16.13. *Let H be as in Theorem* 15.4*, F be a reductive groupoid over $\overline{k}$, and P be a principal (H, F)-bundle. Then the following conditions are equivalent.*

(a) *$\underline{\mathrm{Iso}}_F(P)$ is minimally reductive over H.*
(b) *$H^0_H(X, P \times^F V) = {}^{\mathrm{rad}}H^0_H(X, P \times^F V) \oplus V^F$ for every representation V of F.*

(c) *$P \times^F V$ is a non-trivial indecomposable representation of H for every non-trivial indecomposable representation V of F.*

Proof. It is enough to show that with $K = \underline{\mathrm{Iso}}_F(P)$, each of (b) and (c) is equivalent to the corresponding condition of Corollary 15.6. This is clear because (16.4) is an equivalence and (16.3) with $H = \underline{\mathrm{Iso}}_F(P)$ is an isomorphism. $\square$

Let F be a transitive affine groupoid over an extension k' of k, P be a principal (H, F)-bundle, F_0 be a transitive affine subgroupoid of F, and P_0 be a principal (H, F_0)-subbundle of P. We can form other principal subbundles of P starting from P_0 using the following operations of conjugation, extension and gauge transformation. The conjugate of P_0 by g in $H^0(k', F^{\mathrm{diag}})$ is the principal $(H, g^{-1}F_0g)$-subbundle

$$P_0g$$

of P defined as the image of P_0 under right translation $P \to P$ by g. It is the push forward of P_0 along the morphism of groupoids $F_0 \to g^{-1}F_0g$ defined by $g_0 \mapsto g^{-1}g_0g$. The extension of P_0 to a transitive affine subgroupoid F_1 of F which contains F_0 is the principal (H, F_1)-subbundle

$$P_0F_1$$

of P defined as the image of $P_0 \times_{k'} F_1^{\mathrm{diag}}$ under the action of F on P. It is the push forward of P_0 along $F_0 \to F_1$. The gauge transform of P_0 by the (H, F)-automorphism θ of P is the principal (H, G_0)-subbundle

$$\theta P_0$$

of P defined as the image of P_0 under θ. Two principal (H, F_0)-subbundles of P are gauge transforms of each other if and only if they are (H, F_0)-isomorphic, because such an (H, F_0)-isomorphism extends to an (H, F)-automorphism of P.

Let F' be a transitive affine subgroupoid of F. Then the principal (H, F')-subbundles of P obtained from P_0 by iterating the above three operations are those of the form

$$\theta P_0gF'$$

with $g^{-1}F_0g$ contained in F'. A principal (H, F')-bundle P' can be written in this form for a given g if and only if the morphism of groupoids

$$\alpha_g : F_0 \to F'$$

defined by $g_0 \mapsto g^{-1}g_0g$ sends $[P_0]$ in $H^1_H(X, F_0)$ to $[P']$ in $H^1_H(X, F')$. We note that $\alpha_{\widetilde{g}} = \alpha_g$ if and only if $\widetilde{g} = g_0g$ with g_0 in $H^0_{F_0}(k', F^{\mathrm{diag}})$, and that $\alpha_{\widetilde{g}}$ and α_g are conjugate if and only if $\widetilde{g} = g_0gg'$ with g_0 in $H^0_{F_0}(k', F^{\mathrm{diag}})$ and g' in $H^0(k', F'^{\mathrm{diag}})$.

Corollary 16.14. *Let H be as in Theorem 15.4, G be an affine k-group, and P be a principal (H, G)-bundle. Let G_0 be a reductive k-subgroup of G and P_0 be a principal (H, G_0)-subbundle of P which has no principal (H, G_1)-subbundle for any reductive k-subgroup $G_1 \neq G_0$ of G_0. Let G' be a k-subgroup of G and P' be a principal (H, G')-subbundle of P which has a principal (H, G'')-subbundle for some reductive k-subgroup G'' of G'. Suppose X has a k-point above which both P_0 and P' have a k-point.*

(i) *There is a k-point g of G with gG_0g^{-1} contained in G' for which P' is the image of $P_0g^{-1}G'$ under some (H, G)-automorphism of P.*

(ii) *A g as in* (i) *is unique up to multiplication on the right by a k-point of the centraliser of G_0 in G and on the left by a k-point of G'.*

Proof. Let x be a k-point of X above which P_0 and P' have a k-point. With α_g as above, it is enough to show that there is a g, unique as in (ii), for which α_g sends $[P_0]$ in $\widetilde{H}^1_H(X,x,G_0)$ to $[P_1]$ in $\widetilde{H}^1_H(X,x,G_1)$. The uniqueness is clear by Corollary 16.10.

Let $\widetilde{G}$ and $[\widetilde{P}]$ represent the functor of Theorem 16.5(i). By the condition on P_0, there is a surjective morphism $h_0 : \widetilde{G} \to G_0$ which sends $[\widetilde{P}]$ to $[P_0]$. By the condition on P' and the equivalence (5.5), P' has principal (H,G'')-subbundle with a k-point above x for some reductive k-subgroup G'' of G'. Thus there is a morphism $h' : \widetilde{G} \to G'$ which sends $[\widetilde{P}]$ to $[P']$. If $e_0 : G_0 \to G$ and $e' : G' \to G$ are the embeddings, $e' \circ h'$ is by Corollary 16.10 the conjugate of $e_0 \circ h_0$ by g^{-1} for some k-point g of G. Since h_0 is surjective, $g^{-1}G_0 g$ is contained in G_1 and $h' = \alpha_g \circ h_0$. Thus α_g sends $[P_0]$ to $[P_1]$. □

Corollary 16.15. *Let H be as in Theorem* 15.4*, F be a transitive affine groupoid over $\overline{k}$, and P be a principal (H,F)-bundle. Let F_0 be a reductive subgroupoid of F and P_0 be a principal (H,F_0)-subbundle of P which has no principal (H,F_1)-subbundle for any reductive subgroupoid $F_1 \neq F_0$ of F_0. Let F' be a transitive affine subgroupoid of F and P' be a principal (H,F')-subbundle of P which has a principal (H,F'')-subbundle for some reductive subgroupoid F'' of F'.*

(i) *There is a g in $H^0(\overline{k},F^{\mathrm{diag}})$ with $g^{-1}F_0 g$ contained in F' for which P' is the image of $P_0 gF'$ under some (H,F)-automorphism of P.*
(ii) *A g as in* (i) *is unique up to composition on the left by an element of $H^0_{F_0}(\overline{k},F^{\mathrm{diag}})$ and on the right by an element of $H^0(\overline{k},F'^{\mathrm{diag}})$.*

Proof. With α_g as above, it is enough to show that there is a g, unique as in (ii), for which α_g sends $[P_0]$ in $H^1_H(X,F_0)$ to $[P_1]$ in $H^1_H(X,F_1)$. The uniqueness is clear by Corollary 16.10.

Let $\widetilde{F}$ and $[\widetilde{P}]$ represent the functor of Theorem 16.5(ii). By the condition on P_0, there is a surjective morphism $h_0 : \widetilde{F} \to F_0$ which sends $[\widetilde{P}]$ to $[P_0]$, and by the condition on P', there is a morphism $h' : \widetilde{F} \to F'$ which sends $[\widetilde{P}]$ to $[P']$. If $e_0 : F_0 \to F$ and $e' : F' \to F$ are the embeddings, then $e' \circ h'$ is by Corollary 16.10 the conjugate of $e_0 \circ h_0$ by g^{-1} for some g in $H^0(\overline{k},F^{\mathrm{diag}})$. Since h_0 is surjective, $g^{-1}F_0 g$ is contained in F_1 and $h' = \alpha_g \circ h_0$. Thus α_g sends $[P_0]$ to $[P_1]$. □

Let F be a reductive groupoid over an extension of k, and P be a principal (H,F)-bundle. With $I_H(\mathcal{V})$ as in Section 15, write $I_{H,F}(P)$ for the union of the ideals $I_H(P \times^F V)$ of $H^0_H(X,\mathcal{O}_X)$, where V runs over those representations of F with $V^F = 0$. It is an ideal of $H^0_H(X,\mathcal{O}_X)$, and since (16.4) is an equivalence,

$$I_{H,F}(P) = I_H(\underline{\mathrm{Iso}}_F(P)).$$

The properties of $I_H(K)$ proved in Section 15 thus imply corresponding properties for $I_{H,F}(P)$. In particular by Proposition 15.12, formation of $I_{H,G}(P)$ is compatible with finite extension of scalars and arbitrary extension of scalars when X is quasi-compact, and by Corollary 15.13, $I_{H,F}(P) \neq H^0_H(X,\mathcal{O}_X)$ if and only if $\underline{\mathrm{Iso}}_F(P)$ is minimally reductive over H for H as in Theorem 15.4.

Using Lemmas 16.3, 16.4, 16.6 and 16.7, we obtain analogues for principal bundles of Corollaries 15.16 and 15.18.

By taking for $\overline{k}$ an algebraic closure of k and using Proposition 11.4 and (11.17), the results of Section 15 may also be formulated in terms of principal bundles under a Galois extended $\overline{k}$-group. We state explicitly the following form of Corollary 15.18, which will be required in Sections 19 and 20.

Corollary 16.16. *Let H be as in Theorem* 15.4 *and $\overline{k}$ be an algebraic closure of k. Let (D,E) be a reductive Galois extended $\overline{k}$-group, α be an element of $H^1_H(X,D,E)$ and $\overline{\alpha}$ the image of α in $H^1_H(X_{\overline{k}},D)$. Suppose that X is quasi-compact and quasi-separated and $H_{[1]}$ is quasi-compact. Then (D,E) represents the functor $H^1_H(X,-,-)$ on reductive Galois extended $\overline{k}$-groups up to conjugacy with universal element α if and only if D represents the functor $H^1_H(X_{\overline{k}},-)$ on reductive $\overline{k}$-groups up to conjugacy with universal element $\overline{\alpha}$.*

Proof. By Proposition 11.4, we may suppose that $(D,E) = (F^{\mathrm{diag}}, F(\overline{k})_{\overline{k}})$ for a reductive groupoid F over $\overline{k}$. By (11.17), it is then enough to show that if β in $H^1(X,F)$ has image $\overline{\beta}$ in $H^1(X_{\overline{k}},F^{\mathrm{diag}})$, then F represents $H^1(X,-)$ on reductive groupoids over $\overline{k}$ up to conjugacy with universal element β if and only if F^{diag} represents $H^1(X_{\overline{k}},-)$ on reductive $\overline{k}$-groups up to conjugacy with universal element $\overline{\beta}$. This is clear from Corollary 15.18 and Lemma 16.4, because if β is the class of the principal F-bundle P over X, then $\overline{\beta}$ is the class of the underlying principal F^{diag}-bundle over $X_{\overline{k}}$ of P, while $\underline{\mathrm{Iso}}_{F^{\mathrm{diag}}}(P) = \underline{\mathrm{Iso}}_F(P)_{\overline{k}}$. □

17. Gauge groups

In this section k is a field of characteristic 0*, X is a non-empty k-scheme, and H is a pregroupoid over X.*

In this section we consider the k-groups of cross-sections $\underline{H}^0_H(X,J)$ of certain affine H-groups J. Such k-groups exist for example when X is proper over k and $J = K^{\mathrm{diag}}$ for a transitive affine groupoid over H. In general, $\underline{H}^0_H(X,K^{\mathrm{diag}})$ is not reductive, even if K is. The main result of this section, Theorem 17.5, shows however that if K is reductive and K_0 is a minimal reductive subgroupoid of K, then $\underline{H}^0_{K_0}(X,K^{\mathrm{diag}})$ is a Levi k-subgroup of $\underline{H}^0_H(X,K^{\mathrm{diag}})$. For simplicity, the main results of this section will be given only for H-groups of finite type. The general case is not much more difficult.

Let $\mathcal{V}$ be an H-module and V be a k-vector space. Then we have a homomorphism of k-vector spaces

$$H^0_H(X,\mathcal{V}) \otimes_k V \to H^0_H(X,\mathcal{V} \otimes_k V), \tag{17.1}$$

natural in $\mathcal{V}$ and V, which sends $w \otimes v$ to the image of w under the morphism of H-modules from $\mathcal{V}$ to $\mathcal{V} \otimes_k V$ defined by the embedding of v into V. The homomorphism (17.1) is always injective, and is an isomorphism if and only if every element of its target lies in $H^0_H(X,\mathcal{V} \otimes_k V_0)$ for some finite-dimensional k-vector subspace V_0 of V. It is an isomorphism if V is finite-dimensional over k, or if X is quasi-compact, or if H is a transitive affine groupoid over X. If $\mathcal{V}' \to \mathcal{V}$ is an injective morphism of H-modules and if (17.1) is an isomorphism, then (17.1) with $\mathcal{V}$ replaced by $\mathcal{V}'$ is an isomorphism.

Definition 17.1. Let K be a transitive affine groupoid over H. We say that H is *K-finite* if $H^0_H(X,\mathcal{V})$ is finite-dimensional over k for every representation $\mathcal{V}$ of K and (17.1) is an isomorphism for every representation $\mathcal{V}$ of K and k-vector space V.

If X is proper over k, or if H is a transitive affine groupoid over X, then H is K-finite for every transitive affine groupoid K over H. If H is K-finite, then formation of $H^0_H(X,\mathcal{V})$ for $\mathcal{V}$ a representation of K commutes with extension of scalars.

Let K and K' be transitive affine groupoids over H. If there exists a morphism of groupoids from K to K' over H, then H is K'-finite when it is K-finite. If K is a subgroupoid over H of a transitive affine groupoid K' over H, and K has a reductive subgroupoid over H, then H is K-finite when it is K'-finite: we may suppose that K is reductive, and every representation of K is then a direct summand of a representation of K'.

Let K be a transitive affine groupoid over X and k' be an extension of k. Then every representation $\mathcal{V}'$ of $K_{k'}$ is a quotient of a representation $\mathcal{V}_{k'}$ of $K_{k'}$ for $\mathcal{V}$ a representation of K. Indeed we have a canonical epimorphism of K'-modules $\mathcal{V}'_{k'} \to \mathcal{V}'$, and it suffices to write the K-module $\mathcal{V}'$ as the filtered limit of its finitely generated K-submodules. Taking duals now shows that every representation of $K_{k'}$ is a subrepresentation of a representaion $\mathcal{V}_{k'}$ of $K_{k'}$ for $\mathcal{V}$ a representation of K. It follows from this that if K is a groupoid over H and H is K-finite, then $H_{k'}$ is $K_{k'}$-finite.

Let H be a pregroupoid over X and Z be an H-scheme over X. If the functor $\mathrm{Hom}_H(-_X, Z)$ on k-schemes is representable, we write

$$\underline{H}^0_H(X,Z)$$

for the representing k-scheme. Then we have a functor $\underline{H}^0_H(X,-)$ to the category of k-schemes from the category of those H-schemes Z for which $\underline{H}^0_H(X,Z)$ exists. This last category is closed under the formation of limits, and $\underline{H}^0_H(X,-)$ preserves them. Also $\underline{H}^0_H(X,-)$ is compatible with extension of scalars, and sends monomporphisms of H-schemes to monomorphisms of k-schemes. Any morphism $H \to H'$ of pregroupoids over X induces a monomorphism of k-schemes from $\underline{H}^0_{H'}(X,Z)$ to $\underline{H}^0_H(X,Z)$, when both exist. If $H = X$, then $\underline{H}^0_H(X,Z)$ is the Weil restriction $R_{X/k}Z$.

Proposition 17.2. *Let K be a transitive affine groupoid over H and Z be an affine K-scheme. Suppose that H is K-finite. Then the k-scheme $\underline{H}^0_H(X,Z)$ exists and is affine. It is of finite type over k if Z is of finite type over X.*

Proof. We have $Z = \mathrm{Spec}(\mathcal{R})$ for a commutative K-algebra $\mathcal{R}$. We may write $\mathcal{R}$ as the coequaliser in the category of commutative K-algebras of two morphisms from $\mathrm{Sym}\,\mathcal{V}'$ to $\mathrm{Sym}\,\mathcal{V}$ for K-modules $\mathcal{V}$ and $\mathcal{V}'$, with $\mathcal{V}$ finitely generated if $\mathcal{R}$ is a finitely generated K-algebra. Then Z is the equaliser of two morphisms from $\mathbf{V}(\mathcal{V})$ to $\mathbf{V}(\mathcal{V}')$ of K-schemes, where $\mathbf{V}$ denotes the spectrum of the symmetric algebra. Since $\mathrm{Hom}_H(Y,-)$ preserves limits for every H-scheme Y, it is enough, after writing $\mathcal{V}$ and $\mathcal{V}'$ as the filtered colimit of their finitely generated K-submodules, to show that if $\mathcal{V}$ is a finitely generated K-module then

$$\mathrm{Hom}_H(-_X, \mathbf{V}(\mathcal{V})) \tag{17.2}$$

is representable by an affine k-scheme of finite type.

Using the K-finiteness of H, we have isomorphisms

$$\begin{aligned}\mathrm{Hom}_H(\mathrm{Spec}(R)_X, \mathbf{V}(\mathcal{V})) &\xrightarrow{\sim} \mathrm{Hom}_H(\mathcal{V}, \mathcal{O}_X \otimes_k R) \xrightarrow{\sim} H^0_H(X, \mathcal{V}^\vee \otimes_k R) \xrightarrow{\sim} \\ &\xrightarrow{\sim} H^0_H(X,\mathcal{V}^\vee) \otimes_k R \xrightarrow{\sim} \mathrm{Hom}_k(\mathrm{Spec}(R), \mathbf{V}(H^0_H(X,\mathcal{V}^\vee)^\vee))\end{aligned}$$

which are natural in the commutative k-algebra R. Since (17.2) is a sheaf for the Zariski topology, it is thus represented by the affine k-scheme of finite type $\mathbf{V}(H_H^0(X, \mathcal{V}^\vee)^\vee)$. $\square$

Let M be a k-group of finite type. The adjoint action of M defines a structure of M-algebra on the Lie algebra $\mathfrak{m}$ of M. If $\mathfrak{n}$ is an M-submodule of $\mathfrak{m}$, there exists a unique connected normal k-subgroup N of M with Lie algebra $\mathfrak{n}$. Indeed if V is faithful representation of M, we may take for N the identity component of the k-subgroup of M that acts trivially on $\mathrm{End}_k(V)/\mathfrak{n}$, where $\mathfrak{n}$ is identified with its image in $\mathrm{End}_k(V)$. The uniqueness of N is clear because the intersection of two k-subgroups of M whose Lie algebras coincide has the same Lie algebra.

Let J be an H-group for which $\underline{H}_H^0(X, J)$ exists. Then $\underline{H}_H^0(X, J)$ has a canonical structure of group scheme over k. Suppose that J is smooth over X and $\underline{H}_H^0(X, J)$ is affine and of finite type over k. Then the Lie algebra $\mathrm{Lie}(J)$ of J is a representation of H, and taking points in the k-algebra of dual numbers shows that

$$\mathrm{Lie}(\underline{H}_H^0(X, J)) = H_H^0(X, \mathrm{Lie}(J)).$$

Suppose further that $H_H^0(X, \mathcal{O}_X)$ is a local k-algebra with residue field k. Then the ideal ${}^{\mathrm{rad}}H_H^0(X, \mathrm{Lie}(J))$ of $H_H^0(X, \mathrm{Lie}(J))$ is stable under the adjoint action of $\underline{H}_H^0(X, J)$, as can be seen starting from the fact that any k-point of $\underline{H}_H^0(X, J)$ induces by conjugation an H-morphism $J \to J$. There thus exists a unique connected normal k-subgroup

$${}^{\mathrm{rad}}\underline{H}_H^0(X, J)$$

of $\underline{H}_H^0(X, J)$ with

$$\mathrm{Lie}({}^{\mathrm{rad}}\underline{H}_H^0(X, J)) = {}^{\mathrm{rad}}H_H^0(X, \mathrm{Lie}(J)).$$

It is functorial in J. By Proposition 17.2, the above conditions on H and J are satisfied in particular when X is geometrically H-connected and J is a K-group for some transitive affine groupoid K over H such that H is K-finite.

Let V be a faithful representation of a k-group M of finite type. If we identify the Lie algebra $\mathfrak{m}$ of M with its image in $\mathrm{End}_k(V)$, then the trace on $\mathrm{End}_k(V)$ defines a symmetric pairing

$$a \otimes b \mapsto \mathrm{tr}(a \circ b)$$

on $\mathfrak{m}$. When $\mathfrak{m}$ is equipped with the adjoint action of M, this pairing is a homomorphism of M-modules. Its kernel is the Lie algebra of R_uM.

Proposition 17.3. *Let K be a transitive affine groupoid over H and J be an affine K-group of finite type. Suppose that X is geometrically H-connected and that H is K-finite. Then ${}^{\mathrm{rad}}\underline{H}_H^0(X, J)$ is contained in $R_u\underline{H}_H^0(X, J)$, and coincides with it if the fibres of J are reductive.*

Proof. Since ${}^{\mathrm{rad}}\underline{H}_H^0(X, J)$ and $R_u\underline{H}_H^0(X, J)$ are connected, it is enough to prove that we have an inclusion

$${}^{\mathrm{rad}}H_H^0(X, \mathrm{Lie}(J)) \subset \mathrm{Lie}(R_u\underline{H}_H^0(X, J))$$

of Lie algebras, with equality if the fibres of J are reductive.

Extending the scalars if necessary, we may assume that k is algebraically closed and that X has a k-point x. Since J is of finite type, K acts on J through a quotient K_1 of finite type. Replacing K by K_1, we may assume that K is of finite type. Then $J \rtimes_X K$ is a transitive affine groupoid of finite type over X, and has

thus a faithful representation $\mathcal{W}$. It defines a faithful representation of J on $\mathcal{W}$ which is compatible with the actions of K on J and $\mathcal{W}$.

The action of J on $\mathcal{W}$ defines an embedding

$$\rho : \mathrm{Lie}(J) \to \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{W}).$$

of representations of K. We then have a homomorphism

$$\mathrm{Lie}(J) \to \mathrm{Lie}(J)^{\vee} \tag{17.3}$$

of representations of K which sends the section β of $\mathrm{Lie}(G)$ to $\mathrm{tr}(\rho(-) \circ \rho(\beta))$. Taking the fibre at x and using the above characterisation of the Lie algebra of the unipotent radical shows that (17.3) is an isomorphism if and only if the fibres of J are reductive.

We have a commutative square

$$\begin{array}{ccc} H^0_H(X, \mathrm{Lie}(J)) \otimes H^0_H(X, \mathrm{Lie}(J)^{\vee}) & \longrightarrow & k \\ \big\uparrow & & \big\uparrow \\ H^0_H(X, \mathrm{Lie}(J)) \otimes H^0_H(X, \mathrm{Lie}(J)) & \longrightarrow & H^0_H(X, \mathcal{O}_X) \end{array}$$

where the top arrow is the canonical pairing, the left arrow is defined using (17.3), the bottom arrow sends $\alpha \otimes \beta$ to $\mathrm{tr}(\rho(\alpha) \circ \rho(\beta))$, and the right arrow is the projection onto the residue field. Write

$$M = \underline{H}^0_H(X, J).$$

The Lie algebra of M is then

$$\mathfrak{m} = H^0_H(X, \mathrm{Lie}(J)).$$

If $\mathfrak{n}$ is the kernel of the pairing Q on $\mathfrak{m}$ defined by the square, we show that

$$\mathfrak{n} = \mathrm{Lie}(R_u M).$$

The required results will follow, because the left arrow of the square is an isomorphism if the fibres of J are reductive.

Taking fibres at x, we obtain an action of J_x, and hence of M, on $\mathcal{W}_x$. There is then a filtration of $\mathcal{W}_x$ by M-submodules on whose steps $R_u M$ acts trivially. Given α in $\mathrm{Lie}(R_u M)$ and β in $\mathfrak{m}$, it follows that $\mathrm{tr}(\rho(\alpha) \circ \rho(\beta))$ is 0 at x, and hence is a non-unit in $H^0_H(X, \mathcal{O}_X)$. This proves that $\mathrm{Lie}(R_u M) \subset \mathfrak{n}$.

If we equip $\mathfrak{m}$ with the adjoint action of M and $H^0_H(X, \mathcal{O}_X)$ with the trivial action of M, then the bottom arrow of the square is an M-homomorphism, because the trace of an endomorphism of $\mathcal{W}$ is invariant under conjugation by an automorphism of $\mathcal{W}$. Thus the pairing Q is an M-homomorphism, so that $\mathfrak{n}$ is an M-ideal of $\mathfrak{m}$, and hence is the Lie algebra of a connected normal k-subgroup N of M. We show that every k-homomorphism

$$\mu : \mathbf{G}_m \to M$$

which factors through N is trivial. Since by assumption k is algebraically closed, this will imply that N is unipotent, and hence that $\mathfrak{n} \subset \mathrm{Lie}(R_u M)$.

The k-homomorphism μ is trivial if and only if the homomorphism

$$\widetilde{\mu} : \mathbf{G}_{mX} \to J$$

of group schemes over X that corresponds to it by the universal property of M is trivial. The action of J on $\mathcal{W}$ defines by composition with $\widetilde{\mu}$ an action of $\mathbf{G}_{mX}$ on $\mathcal{W}$, and hence direct sum decomposition of $\mathcal{W}$ into $\mathcal{O}_X$-submodules $\mathcal{W}_n$ on which $\mathbf{G}_{mX}$ acts as the nth power character. Since μ factors through N, the image t

under μ of 1 in the Lie algebra k of $\mathbf{G}_m$ lies in $\mathfrak{n}$. Thus $Q(t \otimes t) = 0$. If d_n in $H^0(X, \mathcal{O}_X)$ is the rank of $\mathcal{W}_n$, it then follows from the definition of Q that

$$\operatorname{tr}(\rho(t) \circ \rho(t)) = \sum_n n^2 d_n$$

is nilpotent in the finite local k-algebra $H^0_H(X, \mathcal{O}_X)$. Hence $d_n = 0$ for $n \neq 0$, so that $\mathbf{G}_{mX}$ acts trivially on $\mathcal{W}$. Since J acts faithfully on $\mathcal{W}$, it follows that $\widetilde{\mu}$ and hence μ is trivial. □

Let H, K and J be as in Proposition 17.3. Then the canonical morphism from $\underline{H}^0_K(X, J)$ to $\underline{H}^0_H(X, J)$ is a monomorphism of k-schemes and hence the embedding of a k-subgroup. Suppose now that K is minimally reductive over H. Then taking $\mathcal{V} = \operatorname{Lie}(J)$ in Corollary 15.6(b) shows that the Lie algebra of $\underline{H}^0_H(X, J)$ is the semidirect product of the Lie algebras of $\underline{H}^0_K(X, J)$ and ${}^{\mathrm{rad}}\underline{H}^0_H(X, J)$. The more precise result that we have a semidirect product decomposition for the k-groups themselves is proved in Theorem 17.5 below. In particular, it follows from Proposition 17.3 and Theorem 17.5 that if the fibres of J are reductive then $\underline{H}^0_K(X, J)$ is a Levi k-subgroup of $\underline{H}^0_H(X, J)$. The proof of Theorem 17.5 is based on the characterisation of the k-points of ${}^{\mathrm{rad}}\underline{H}^0_H(X, J)$ given by Lemma 17.4 below in the case where $J = L^{\mathrm{diag}}$ where L is a transitive affine groupoid of finite type over H which has a reductive subgroupoid over H.

Let L be a transitive affine groupoid over H. Then by Lemma 14.9 with $K = L$ and $f_1 = f_2$ the structural morphism of H, we have a group isomorphism

$$H^0_H(X, L^{\mathrm{diag}}) \xrightarrow{\sim} \operatorname{Aut}^{\otimes}(\omega_{L/H}) \tag{17.4}$$

whose composite with the homomorphism that sends a tensor automorphism of $\omega_{L/H}$ to its component at the representation $\mathcal{V}$ of L is the homomorphism

$$H^0_H(X, L^{\mathrm{diag}}) \to H^0_H(X, \underline{\operatorname{Aut}}_{\mathcal{O}_X}(\mathcal{V})) = \operatorname{Aut}_H(\mathcal{V}) \tag{17.5}$$

induced by the action $L^{\mathrm{diag}} \to \underline{\operatorname{Aut}}_{\mathcal{O}_X}(\mathcal{V})$ of L^{diag} on $\mathcal{V}$.

Suppose that $H^0_H(X, \mathcal{O}_X)$ is a local k-algebra with residue field k. If

$$Q : \operatorname{Mod}_H(X) \to \overline{\operatorname{Mod}_H(X)}$$

is the projection, then we have a group homomorphism

$$\operatorname{Aut}^{\otimes}(\omega_{L/H}) \to \operatorname{Aut}^{\otimes}(Q\omega_{L/H}) \tag{17.6}$$

defined by composition with Q.

Lemma 17.4. *Let L be a transitive affine groupoid of finite type over H. Suppose that X is geometrically H-connected and that H is L-finite. Then the group of k-points of ${}^{\mathrm{rad}}\underline{H}^0_H(X, L^{\mathrm{diag}})$ is contained in the kernel of the composite of* (17.4) *with* (17.6)*, and coincides with this kernel if L has a reductive subgroupoid over H.*

Proof. Let $\mathcal{V}$ be a representation of L. Then the k-group of automorphisms

$$\underline{\operatorname{Aut}}_H(\mathcal{V}) = \underline{H}^0_H(X, \underline{\operatorname{Aut}}_{\mathcal{O}_X}(\mathcal{V}))$$

exists, and it is the k-group of units of the finite k-algebra

$$\operatorname{End}_H(\mathcal{V}) = H^0_H(X, \underline{\operatorname{End}}_{\mathcal{O}_X}(\mathcal{V}))$$

because the appropriate homomorphisms (17.1) are isomorphisms. The action of L^{diag} on $\mathcal{V}$ defines a k-homomorphism

$$\eta_{\mathcal{V}} : \underline{H}^0_H(X, L^{\mathrm{diag}}) \to \underline{H}^0_H(X, \underline{\mathrm{Aut}}_{\mathcal{O}_X}(\mathcal{V})),$$

which induces on k-points the homomorphism (17.5). Now the kernel $\mathrm{Ker}(Q_{\mathcal{V},\mathcal{V}})$ of Q on $\mathrm{End}_H(\mathcal{V})$ is a nilideal, because each of its elements has trace 0. It follows that $\underline{\mathrm{Aut}}_H(\mathcal{V})$ has a unipotent k-subgroup $U_{\mathcal{V}}$ such that

$$\mathrm{Lie}(U_{\mathcal{V}}) = \mathrm{Ker}(Q_{\mathcal{V},\mathcal{V}}) = {}^{\mathrm{rad}}H^0_H(X, \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{V}))$$

and such that the k-points of $U_{\mathcal{V}}$ are those H-automorphism of $\mathcal{V}$ lying above the identity in $\overline{\mathrm{Mod}_H(X)}$. The kernel of the composite of (17.4) with (17.6) then consists of those k-points of $\underline{H}^0_H(X, L^{\mathrm{diag}})$ which lie in ${\eta_{\mathcal{V}}}^{-1}(U_{\mathcal{V}})$ for every $\mathcal{V}$.

The action of $\mathrm{Lie}(L^{\mathrm{diag}})$ on $\mathcal{V}$ defines a homomorphism of Lie algebras

$$h_{\mathcal{V}} : H^0_H(X, \mathrm{Lie}(L^{\mathrm{diag}})) \to H^0_H(X, \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{V})),$$

which coincides with that induced by $\eta_{\mathcal{V}}$. By functoriality of ${}^{\mathrm{rad}}H^0_H(X, -)$ we have

$$(17.7) \qquad {}^{\mathrm{rad}}H^0_H(X, \mathrm{Lie}(L^{\mathrm{diag}})) \subset {h_{\mathcal{V}}}^{-1}({}^{\mathrm{rad}}H^0_H(X, \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{V}))).$$

Since ${}^{\mathrm{rad}}\underline{H}^0_H(X, L^{\mathrm{diag}})$ is unipotent and hence connected, this implies that

$$(17.8) \qquad {}^{\mathrm{rad}}\underline{H}^0_H(X, L^{\mathrm{diag}}) \subset {\eta_{\mathcal{V}}}^{-1}(U_{\mathcal{V}}).$$

The required inclusion of k-points follows.

Suppose now that L has a reductive subgroupoid over H. Let $\mathcal{V}$ be a faithful representation of L, and consider the commutative square (9.3) with $\mathcal{V}$ replaced by $\mathrm{Lie}(L^{\mathrm{diag}})$ and $\mathcal{V}'$ by $\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{V})$, and f the action of $\mathrm{Lie}(L^{\mathrm{diag}})$ on $\mathcal{V}$. The action f is a monomorphism in $\mathrm{Mod}_L(X)$, so that $f^\vee$ is an epimorphism in $\mathrm{Mod}_L(X)$ and hence a retraction in $\mathrm{Mod}_H(X)$. In the top arrow of (9.3), the homomorphism induced by $f^\vee$ is thus surjective, so that we have equality in (17.7). Hence we have equality in (17.8), because $\eta_{\mathcal{V}}$ is an embedding, so that ${\eta_{\mathcal{V}}}^{-1}(U_{\mathcal{V}})$ is unipotent and hence connected. The required equality of k-points follows. □

Theorem 17.5. *Let K be a minimal reductive groupoid over H and J be an affine K-group of finite type. Suppose that X is geometrically H-connected and H is K-finite. Then $\underline{H}^0_H(X, J)$ is the semidirect product of its k-subgroup $\underline{H}^0_K(X, J)$ with its normal k-subgroup ${}^{\mathrm{rad}}\underline{H}^0_H(X, J)$.*

Proof. Replacing K by a quotient through which it acts on J, we may suppose that K is of finite type. It is enough to prove the particular case where $J = L^{\mathrm{diag}}$ for a transitive affine groupoid L over K of finite type: the general case follows by taking $L = J \rtimes_X K$, because the semidirect product decomposition $L^{\mathrm{diag}} = J \rtimes_X K^{\mathrm{diag}}$ is preserved by each of $\underline{H}^0_H(X, -)$, $\underline{H}^0_K(X, -)$ and ${}^{\mathrm{rad}}\underline{H}^0_H(X, -)$.

After extending the scalars, we may assume that k is algebraically closed. It is then enough to show that we have a semidirect product decomposition on groups of k-points. We have a commutative diagram

$$\begin{array}{ccccc}
H^0_K(X, L^{\mathrm{diag}}) & \longrightarrow & H^0_H(X, L^{\mathrm{diag}}) & \longrightarrow & \mathrm{Aut}^{\otimes}(Q\omega_{L/H}) \\
\downarrow \wr & & \downarrow \wr & & \| \\
\mathrm{Aut}^{\otimes}(\omega_{L/K}) & \longrightarrow & \mathrm{Aut}^{\otimes}(\omega_{L/H}) & \longrightarrow & \mathrm{Aut}^{\otimes}(Q\omega_{L/H})
\end{array}$$

where the vertical isomorphisms are of the form (17.4), the top left arrow is the embedding, the top right arrow is the composite of Lemma 17.4, the bottom left arrow sends φ to $\omega_{K/H}\varphi$, and the bottom right arrow is (17.6). The composite of the bottom and hence of the top arrows is an isomorphism, because $Q\omega_{K/H}$ is fully faithful by Theorem 15.4(ii). Since L has the reductive subgroupoid K over H, the required semidirect product decomposition now follows from Lemma 17.4. $\square$

18. Pullback of universal and minimal reductive groupoids

In this section k is a field of characteristic 0, X is a non-empty k-scheme, and H is a pregroupoid over X.

In this section we study the behaviour of a universal or minimal reductive groupoid K over H under pullback, and under formation of groupoids $K \times_X X'$ for a K-scheme X'. In particular Proposition 18.5, which will be needed in Section 20, shows that if X' is K-proétale, then under apropriate conditions, $K \times_X X'$ is universally reductive over $H \times_X X'$ if and only if K is universally reductive over H.

Proposition 18.1. *Let K be a transitive affine groupoid over H, and X' be a non-empty scheme over X. Suppose that either H is a transitive groupoid or that the structural morphism of X' is fpqc covering. Then K is minimally (resp. universally) reductive over H if and only if $K \times_{[X]} [X']$ is minimally (resp. universally) reductive over $H \times_{[X]} [X']$.*

Proof. Immediate from the fact that $- \times_{[X]} [X']$ induces an equivalence from ttransitive affine groupoids over H to transitive affine groupoids over $H \times_{[X]} [X']$. $\square$

Call a transitive affine groupoid K over a pregroupoid H *almost minimally reductive* if K is reductive and any reductive subgroupoid over H of K contains the connected component K^{con} of K^{diag}. Then K is almost minimally reductive over H if and only every quotient K'' of K of finite type is almost minimally reductive over H. Further K is minimally reductive over H if and only if K is almost minimally reductive over H and $K_{\text{ét}}$ is minimally reductive over H. If X is geometrically H-connected and K' is a transitive affine subgroupoid over H of K containing K^{con}, then K is almost minimally reductive if and only if K' is.

Lemma 18.2. *Let K be a reductive groupoid of finite type over X and K' be a reductive subgroupoid of X. Then the following conditions are equivalent.*

(a) $K'^{\mathrm{con}} = K^{\mathrm{con}}$.
(b) *There are only finitely many pairwise non-isomorphic irreducible representations $\mathcal{V}$ of K for which $H^0_{K'}(X, \mathcal{V}) \neq 0$.*

Proof. Replacing k by a sufficiently large extension k', we may suppose by Lemma 14.12 with $X' = X_{k'}$ that k is algebraically closed and that X has a k-point. By Lemma 3.2, we may further suppose that $X = \mathrm{Spec}(k)$. Then K is a reductive k-group G of finite type and K' is a reductive k-subgroup G' of G.

The k-group G acts by left and right translation on the k-algebra $k[G]$. Both (a) and (b) are equivalent to the condition that the k-algebra $k[G]^{G'}$ of invariants under right translation by G' be finite: for (a) this is so because G/G' is affine, and for (b) by the canonical decomposition of $k[G]$ as a (G, G)-bimodule. $\square$

Proposition 18.3. *Let H be a as in Theorem* 15.4*, K be a transitive affine groupoid over H, and X' be a finite locally free H-scheme which is non-empty and geometrically H-connected. Then $K \times_{[X]} [X']$ is almost minimally reductive over $H \times_X X'$ if and only if K is almost minimally reductive over H.*

Proof. Write H' for $H \times_X X'$ and K' for $K \times_{[X]} [X']$. That K is almost minimally reductive over H if K' is almost minimally reductive over H' is clear.

To prove the converse, we may suppose that K is of finite type. Replacing K by the inverse image of an appropriate subgroupoid of $K_{\text{ét}}$ along the projection $K \to K_{\text{ét}}$, we may suppose further that K is minimally reductive over H. We have

$$X' = \operatorname{Spec}(\mathcal{R})$$

for a finite locally free H-algebra $\mathcal{R}$. If L is a universal reductive groupoid over H, there is a surjective morphism of groupoids $L \to K$ over H, and $\mathcal{R}$ is isomorphic as a representation of H to a representation of L. Replacing K by a sufficiently large quotient of finite type of L through which $L \to K$ factors, we may suppose that $\mathcal{R}$ is isomorphic as a representation of H to a representation $\mathcal{W}$ of K.

Let K'' be a reductive subgroupoid over H' of K', and write $\mathcal{W}'$ for the pullback of $\mathcal{W}$ onto X'. We show that if $\mathcal{V}'$ is an irreducible representation of K' with

$$H^1_{K''}(X', \mathcal{V}') \neq 0,$$

then $\mathcal{V}'$ is a direct summand of

$$\mathcal{W}' \otimes_{\mathcal{O}_{X'}} \mathcal{W}'^{\vee}.$$

It will follow that (b) and hence (a) of Lemma 18.2 is satisfied with K' and K'' for K and K', so that K' is almost minimally reductive over H'.

There is by Lemma 3.2 a representation $\mathcal{V}$ of K with pullback onto X' isomorphic to $\mathcal{V}'$. By hypothesis, $\mathcal{V}'$ has as a representation of K'' and hence as a representation of H' the direct summand $\mathcal{O}_{X'}$. Thus by the equivalence (8.6), $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{V}$ has as a representation of $(H, \mathcal{R})$ and hence as a representation of H the direct summand $\mathcal{R}$. By Proposition 15.3 and (c) of Corollary 15.6, the representation $\mathcal{W} \otimes_{\mathcal{O}_X} \mathcal{V}$ of K has thus the direct summand $\mathcal{W}$ and hence the representation $\mathcal{W}' \otimes_{\mathcal{O}_{X'}} \mathcal{V}'$ of K' has the direct summand $\mathcal{W}'$. Since $\mathcal{V}'$ is irreducible, it is therefore a direct summand of $\mathcal{W}' \otimes_{\mathcal{O}_{X'}} \mathcal{W}'^{\vee}$. $\square$

Proposition 18.4. *Let H be as in Theorem* 15.4*, K be a transitive affine groupoid over H, and X' be a non-empty geometrically H-connected K-scheme. Suppose that one of the following conditions holds.*

(a) *X' is a finite étale K-scheme.*
(b) *X is quasi-compact and X' is a K-proétale K-scheme.*

Then $K \times_X X'$ is minimally reductive over $H \times_X X'$ if and only if K is minimally reductive over H.

Proof. Write H' and K' for $H \times_X X'$ and $K \times_X X'$. Clearly K' is reductive if and only if K is. Suppose that K' is minimally reductive over H'. If L is a reductive subgroupoid of K over H, then $L \times_X X'$ is a reductive subgroupoid of K' over H'. Hence $L \times_X X' = K'$, so that $L = K$ because $X' \to X$ is an *fpqc* covering morphism. Thus K is minimally reductive over H.

Conversely suppose that K is minimally reductive over H. To prove that K' is minimally reductive over H', write $X' = \operatorname{Spec}(\mathcal{R})$ for a K-algebra $\mathcal{R}$. If (a) holds,

$\mathcal{R}$ is finite étale. If (b) holds, $\mathcal{R}$ is the colimit of a filtered system $(\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ of finite étale K-algebras, and we may assume by Lemma 9.3 that $\mathcal{R}$ is faithfully flat over each $\mathcal{R}_\lambda$. The k-algebras $H^0_H(X,\mathcal{R})$ and $H^0_H(X,\mathcal{R}_\lambda)$ are henselian local with residue field k.

By (b) of Corollary 15.6 and the equivalence (8.6), it is enough to show that

$$H^0_{H'}(X',\widetilde{\mathcal{V}}) = {}^{\mathrm{rad}}H^0_{H'}(X',\widetilde{\mathcal{V}}) \oplus H^0_{K'}(X',\widetilde{\mathcal{V}}) \tag{18.1}$$

for every representation $\mathcal{V}$ of $(K,\mathcal{R})$. If (a) holds, this is clear from (b) of Corollary 15.6 and Proposition 9.2.

Assume (b). By the equivalence (8.8) and Lemma 14.12, it is enough to prove (18.1) for $\mathcal{V}$ of the form $\mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{V}_0$ with $\mathcal{V}_0$ a representation of K. By (8.6) and the definition of ${}^{\mathrm{rad}}H$ as the kernel of a pairing (9.2), it is the same to prove that $H^0_H(X,\mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{V}_0)$ is the direct sum of $H^0_K(X,\mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{V}_0)$ and the kernel on the right of the pairing

$$\mathrm{Hom}_{H,\mathcal{R}}(\mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{V}_0,\mathcal{R})\otimes_k H^0_H(X,\mathcal{R}\otimes_{\mathcal{O}_X}\mathcal{V}_0)\to H^0_H(X,\mathcal{R})\to k$$

with the first arrow given by composition and the second by projection onto the residue field. By the case where (a) holds we have such a direct sum decomposition with $\mathcal{R}$ replaced by $\mathcal{R}_\lambda$. It thus suffices to apply Lemma 9.5. □

Suppose that X is geometrically H-connected. Then by Proposition 10.2, the category of proétale groupoids up to H has an initial object $\pi_{H,\text{ét}}(X)$. Let K be a universal reductive groupoid over H. Then K/K^{con} and $\pi_{H,\text{ét}}(X)$ are both initial in the category of proétale groupoids over H up to conjugacy, so that they are isomorphic over H. It follows that there is a unique morphism from K over H to any proétale groupoid over H, and that $K\to\pi_{H,\text{ét}}(X)$ factors through an isomorphism

$$K/K^{\mathrm{con}}\xrightarrow{\sim}\pi_{H,\text{ét}}(X). \tag{18.2}$$

By Lemma 10.6(ii), restriction from K to H thus defines an equivalence from the category of K-proétale K-schemes to the category of H-proétale H-schemes.

Proposition 18.5. *Let H be as in Theorem* 15.4*, K be a transitive affine groupoid over H, and X' be a non-empty geometrically H-connected K-scheme. Suppose that* one *of the following conditions holds.*

(a) *X' a is finite étale K-scheme.*
(b) *X is quasi-compact and quasi-separated, $H_{[1]}$ is quasi-compact, and X' is a K-proétale K-scheme.*

Then $K\times_X X'$ is universally reductive over $H\times_X X'$ if and only if K is universally reductive over H.

Proof. By (c) of Corollary 15.5 and Proposition 18.4, it suffices to apply Lemma 9.7 with $H'=K$. □

Remark 18.6. It is possible to deduce Proposition 18.5 in the case where (a) holds directly from the universal property, using a suitable notion of Weil restriction for groupoids. This does not require any condition on $H^0_H(X,\mathcal{O}_X)$. However it is not clear how to pass from this case to that where X' is an arbitrary filtered limit of finite étale H-schemes. It is also possible to deduce Proposition 18.4 from Proposition 18.3.

If $p : X' \to X$ is a quasi-compact quasi-separated morphism with $p_*\mathcal{O}_{X'} = \mathcal{O}_X$, then the pullback of p along any flat morphism is schematically dominant. For such a p, the condition on H in Proposition 18.7 below is thus satisfied for any H over X for which one of the structural morphisms $H_{[1]} \to X$ is flat.

Proposition 18.7. *Let K be a transitive affine groupoid over H, and X' be a non-empty H-scheme with structural morphism $p : X' \to X$. Suppose that $\mathcal{O}_X = p_*\mathcal{O}_{X'}$ and that the pullbacks of p along the $d_i : H_{[1]} \to X$ factor though no closed subscheme of $H_{[1]}$ other than $H_{[1]}$. Then pullback along $[p]$ induces a bijection between from the set of reductive subgroupoids of K over H to the set of reductive subgroupoids of $K \times_{[X]} [X']$ over $H \times_X X'$.*

Proof. Write H' for $H \times_X X'$ and K' for $K \times_{[X]} [X']$, and let L' be a reductive subgroupoid of K' over H'. It is to be shown that

$$L' = L \times_{[X]} [X'] \subset K' \tag{18.3}$$

for a reductive subgroupoid L over H of K. The quotient K'/L' exists and is a transitive affine K'-scheme, because L' is reductive. We then have

$$K'/L' = Z \times_X X'$$

for a transitive affine K-scheme Z. Since the base cross-section of K'/L' is an H'-morphism, its component $s' : X' \to Z$ at Z is an H-morphism. There is a unique cross section s of Z over X such that s' factors as

$$X' \xrightarrow{p} X \xrightarrow{s} Z,$$

because $\mathcal{O}_X = p_*\mathcal{O}_{X'}$. Then (18.3) holds with L the stabiliser K_s of s. It remains to show that s is an H-morphism, so that K_s is a subgroupoid over H of K.

The condition for s to be an H-morphism is that the composite a of $H_{[1]} \times_{d_1,X} s$ with the isomorphism over $H_{[1]}$ defining the H-structure of Z should coincide with $b = H_{[1]} \times_{d_0,X} s$. Since p and $s \circ p$ are H-morphisms, the composites of $H_{[1]} \times_{d_1,X} p$ with a and b coincide. Thus a and b coincide, because their equaliser is a closed subscheme of $H_{[1]}$ through which $H_{[1]} \times_{d_1,X} p$ factors. □

Corollary 18.8. *For H, K and X' as in Proposition* 18.7*, K is minimally reductive over H if and only if $K \times_{[X]} [X']$ is minimally reductive over $H \times_X X'$.*

Proof. Immediate from the Proposition. □

19. Curves of genus 0

In this section k is a field of characteristic 0 *and $\overline{k}$ is an algebraic closure of k.*

In this section we describe explicitly principal bundles with reductive structure group over a smooth projective curve X over k of genus 0. The study of such bundles is reduced by Theorem 19.1 below to the study of the universal groupoid $\pi_{\mathrm{mult}}(X)$ of multiplicative type of Section 10. The main result is Theorem 19.3. Theorem 19.2 is the well-known algebraic version of Grothendieck's original classication over the Riemann sphere.

Throughout this section, we suppose that X is a k-scheme which satisfies the following condition: it is *reduced, quasi-compact and quasi-separated, with*

$$H^0(X, \mathcal{O}_X) = k. \tag{19.1}$$

This condition is stable under extension of scalars. It is also stable under passage to geometrically connected proétale covers of X, because if X' is finite étale over X, then by the Cayley–Hamilton theorem any element of $H^0(X', \mathcal{O}_{X'})$ is finite over k.

We begin by developing the properties of the groupoid $\pi_{\mathrm{mult}}(X)$ that will be required. It will be convenient to consider at the same time the groupoids $\pi_{\text{ét}}(X)$ and $\pi_{\text{étm}}(X)$, which will be required for the next section.

By Proposition 10.2 (resp. 10.3, resp.10.4) an initial object exists in the category of proétale groupoids (resp. groupoids of multiplicative type, resp. groupoids of proétale by multiplicative type) over X. We write it as $\pi_{\text{ét}}(X)$ (resp. $\pi_{\mathrm{mult}}(X)$, resp. $\pi_{\text{étm}}(X)$). Each of $\pi_{\text{ét}}(X)$, $\pi_{\mathrm{mult}}(X)$ and $\pi_{\text{étm}}(X)$ is functorial in X in the sense that for example given $f : X \to X'$ there is a unique f_* such that

$$(f, f_*) : (X, \pi_{\text{ét}}(X)) \to (X', \pi_{\text{ét}}(X'))$$

is a morphism of groupoids in k-schemes, by the universal property of $\pi_{\text{ét}}(X)$ applied to the pullback of $\pi_{\text{ét}}(X')$ onto X. Further $\pi_{\text{ét}}(X)$ coincides with $(\pi_{\text{étm}}(X))_{\text{ét}}$ and $\pi_{\mathrm{mult}}(X)$ with $(\pi_{\text{étm}}(X))_{\mathrm{mult}}$. A universal reductive groupoid L over X exists by Theorem 15.4(i), and $\pi_{\text{ét}}(X)$, $\pi_{\mathrm{mult}}(X)$ and $\pi_{\text{étm}}(X)$ coincide with $L_{\text{ét}}$, L_{mult} and $L_{\text{étm}}$.

By Corollary 15.18, formation of $\pi_{\text{ét}}(X)$, $\pi_{\mathrm{mult}}(X)$ and $\pi_{\text{étm}}(X)$ commutes with algebraic extension of scalars. Let X' be a proétale cover of X. By Lemma 10.6(ii), X' has a unique structure of $\pi_{\text{ét}}(X)$-scheme and hence of $\pi_{\text{étm}}(X)$-scheme. If X' is geometrically connected, then

$$\pi_{\text{étm}}(X') = \pi_{\text{étm}}(X) \times_X X', \tag{19.2}$$

by Proposition 18.5.

Theorem 19.1. *Let X be a smooth geometrically connected projective curve of genus 0 over k. Then $\pi_{\mathrm{mult}}(X)$ is universally reductive over X.*

Proof. By Corollary 15.16, we may suppose that k is algebraically closed. The indecomposable representations of $\pi_{\mathrm{mult}}(X)$ are then those of rank 1, and passage to the underlying line bundle defines an isomorphism from the group of isomorphism classes such representations to $\mathrm{Pic}(X)$. Condition (b) of Corollary 15.5 is thus satisfied, because the indecomposable vector bundles over X are those of rank 1. $\square$

Let x be a k-point of X. Given an affine k-group G, we denote by

$$H^1(X, x, G)$$

the set of isomorphism classes of pairs (P, p) with P a principal G-bundle over X and p a k-point of P above x. It is pointed set which is contravariant in (X, x) and covariant in G. The functor $H^1(X, x, -)$ does not in general factor through the category of affine k-groups up to conjugacy: the action of $G(k)$ on $H^1(X, x, G)$ through its action by conjugation on G, or equivalently by shifting the k-point p of P, is in general non-trivial. Discarding p defines a bijection

$$H^1(X, x, G)/G(k) \xrightarrow{\sim} \widetilde{H}^1(X, x, G) \tag{19.3}$$

onto the subset of $H^1(X, G)$ consisting of the $[P]$ with P trivial above x. When G is commutative, $G(k)$ acts trivially and $H^1(X, x, G)$ and $\widetilde{H}^1(X, x, G)$ coincide. We

have

$$H^1(X,x,\mathbf{G}_m) = \widetilde{H}^1(X,x,\mathbf{G}_m) = H^1(X,\mathbf{G}_m), \tag{19.4}$$

by Hilbert's Theorem 90, and similarly with $\mathbf{G}_m$ replaced by $\mathbf{G}_a$.

We write $\pi_{\text{ét}}(X,x)$, $\pi_{\text{mult}}(X,x)$ and $\pi_{\text{étm}}(X,x)$ for the respective fibres $\pi_{\text{ét}}(X)_{x,x}$, $\pi_{\text{mult}}(X)_{x,x}$ and $\pi_{\text{étm}}(X)_{x,x}$ above the diagonal. The k-groups $\pi_{\text{ét}}(X,x)$, $\pi_{\text{mult}}(X,x)$ and $\pi_{\text{étm}}(X,x)$ are functorial in (X,x), because $\pi_{\text{ét}}(X)$, $\pi_{\text{mult}}(X)$ and $\pi_{\text{étm}}(X)$ are functorial in X. Since $\pi_{\text{mult}}(X)$ is a commutative groupoid, its diagonal is constant, so that $\pi_{\text{mult}}(X,x)$ is independent of x. By Corollary 15.18, formation of $\pi_{\text{ét}}(X,x)$, $\pi_{\text{mult}}(X,x)$ and $\pi_{\text{étm}}(X,x)$, commutes with algebraic extension of scalars.

By the equivalence (5.5) and the isomorphism (5.6), the triple

$$(\pi_{\text{étm}}(X,x), \pi_{\text{étm}}(X)_{-,x}, 1_x)$$

is initial in the category of k-pointed principal bundles over (X,x) under a k-group of proétale by multiplicative type. Thus the functor $H^1(X,x,-)$ on k-groups of proétale by multiplicative type is represented by $\pi_{\text{étm}}(X,x)$, with the universal element in $H^1(X,x,\pi_{\text{étm}}(X,x))$ the class of $(\pi_{\text{étm}}(X)_{-,x},1_x)$.

The morphisms of k-groups $\pi_{\text{étm}}(X,x)$ defined by functoriality are those compatible with the universal elements, so that given a morphism f from $(X.x)$ to (X',x'), we have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_k(\pi_{\text{étm}}(X',x'),G) \arrow[r, "\sim"] \arrow[d, "\mathrm{Hom}_k(f_*,G)"'] & H^1(X',x',G) \arrow[d, "f^*"] \\
\mathrm{Hom}_k(\pi_{\text{étm}}(X,x),G) \arrow[r, "\sim"] & H^1(X,x,G)
\end{tikzcd} \tag{19.5}$$

for G of proétale by multiplicative type, where the horizontal arrows are the natural bijections defined by the universal elements.

There are similar universal properties for $\pi_{\text{ét}}(X,x)$ and $\pi_{\text{mult}}(X,x)$. The universal elements in $H^1(X,x,\pi_{\text{ét}}(X,x))$ and $H^1(X,x,\pi_{\text{mult}}(X,x))$ are the images of the universal element in $H^1(X,x,\pi_{\text{étm}}(X,x))$, and the analogues (19.5) hold.

The k-pointed geometric universal cover $(\widetilde{X},\widetilde{x})$ of (X,x) may be identified with $(\pi_{\text{ét}}(X)_{-,x},1_x)$. It is functorisl in (X,x), and its formation commutes with algebraic extension of scalars. Taking $\widetilde{X}$ for X' in (19.2) shows that we have a short exact sequence of k-groups

$$1 \to \pi_{\text{mult}}(\widetilde{X},\widetilde{x}) \to \pi_{\text{étm}}(X,x) \to \pi_{\text{ét}}(X,x) \to 1, \tag{19.6}$$

functorial in (X,x), with $\pi_{\text{étm}}(\widetilde{X},\widetilde{x}) = \pi_{\text{mult}}(\widetilde{X},\widetilde{x})$ connected because $\widetilde{X}$ is geometrically simply connected.

For G a k-group of proétale by multiplicative type, we have by the universal property of $\pi_{\text{étm}}(X,x)$ and the compatibility of $\pi_{\text{étm}}(X)$ with algebraic extension of scalars a canonical bijection

$$H^1(X,x,G) \xrightarrow{\sim} H^1(X_{\overline{k}},\overline{x},G_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$$

with $\overline{x}$ the $\overline{k}$-point of $X_{\overline{k}}$ defined by x. When $G = \pi_{\text{étm}}(X,x)$, this bijection respects the universal elements. The bijection becomes

$$H^1(X,x,G) \xrightarrow{\sim} H^1(X_{\overline{k}},G_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$$

for G of multiplicative type.

Let M be a Galois module, i.e. an abelian group with a continuous action of $\mathrm{Gal}(\overline{k}/k)$. We write

$$D(M)$$

for the k-group of multiplicative type with Galois module of characters M, and

$$\chi_\mu : D(M)_{\overline{k}} \to (\mathbf{G}_m)_{\overline{k}}$$

for the character corresponding to $\mu \in M$.

By the universal property of $\pi_{\mathrm{mult}}(X, x)$, the compatibility of $\pi_{\mathrm{mult}}(X)$ with algebraic extension of scalars, and (19.4), the Galois module of characters of $\pi_{\mathrm{mult}}(X, x)$ is

$$\mathrm{Pic}(X_{\overline{k}}) = H^1(X_{\overline{k}}, (\mathbf{G}_m)_{\overline{k}}).$$

We then have

$$\pi_{\mathrm{mult}}(X, x) = D(\mathrm{Pic}(X_{\overline{k}}))$$

with the universal element in $H^1(X, x, D(\mathrm{Pic}(X_{\overline{k}})))$ uniquely determined by the condition that its image under the homomorphism to $H^1(X_{\overline{k}}, (\mathbf{G}_m)_{\overline{k}})$ defined by χ_μ is μ.

The assignment to every line bundle of its degree identifies $\mathrm{Pic}(\mathbf{P}^n_{\overline{k}})$ with the trivial Galois module $\mathbf{Z}$. Thus

$$\pi_{\mathrm{mult}}(\mathbf{P}^n, x) = \mathbf{G}_m$$

for every n and k-point x of $\mathbf{P}^n$. By (19.4), we have

$$H^1(\mathbf{P}^n, x, \mathbf{G}_m) = \widetilde{H}^1(\mathbf{P}^n, x, \mathbf{G}_m) = H^1(\mathbf{P}^n, \mathbf{G}_m) = \mathbf{Z},$$

with the universal element of degree 1.

Theorem 19.2. *Let x be a k-point of $\mathbf{P}^1$. Then the functor $\widetilde{H}^1(\mathbf{P}^1, x, -)$ on reductive k-groups up to conjugacy is represented by $\mathbf{G}_m$, with universal element in*

$$\widetilde{H}^1(\mathbf{P}^1, x, \mathbf{G}_m) = H^1(\mathbf{P}^1, \mathbf{G}_m) = \mathbf{Z}$$

the class of degree 1.

Proof. With the identification $\pi_{\mathrm{mult}}(\mathbf{P}^1, x) = \mathbf{G}_m$, a principal $\mathbf{G}_m$-bundle P over X of degree 1 is isomorphic to $\pi_{\mathrm{mult}}(\mathbf{P}^1)_{-,x}$. Thus by (5.6), $\underline{\mathrm{Iso}}_{\mathbf{G}_m}(P)$ is isomorphic to $\pi_{\mathrm{mult}}(\mathbf{P}^1)$, and hence by Theorem 19.1 is universally reductive over $\mathbf{P}^1$. The result now follows from Lemma 16.3. □

We continue to assume that (19.1) and the other conditions on X hold, but we no longer assume that X has a k-point. In that case $\pi_{\mathrm{mult}}(X)$ need not arise from a principal bundle under an affine k-group, and it is necessary to consider groupoids over $\overline{k}$ and Galois extended $\overline{k}$-groups. Before doing this, it will convenient to recall first some definitions concerning continuous cohomology groups.

Let M be a topological group and N be a topological group with an action of M (by group automorphisms). Write $H^0(M, N)$ for the subgroup of N of invariants under M, and $H^1(M, N)$ for the pointed set of orbits under the action by conjugation of N of the set of sections $M \to N \rtimes M$ of topological groups to the projection onto M. Both $H^0(M, N)$ and $H^1(M, N)$ are functorial in N. When M is profinite and N is discrete, $H^0(M, N)$ is the usual group and $H^1(M, N)$ is the usual pointed set.

Let N and N' be topological groups. We write $\mathcal{E}(N, N')$ for the set of isomorphism classes of extensions of N by N'. Given an extension E' of N by N' and an

isomorphism $i : N' \xrightarrow{\sim} N''$ of topological groups, a pair consisting of an extension E'' of N by N'' and an isomorphism from E' to E'' with component the 1_N at N and i at N' is unique up to unique isomorphism. Thus i induces a map from $\mathcal{E}(N, N')$ to $\mathcal{E}(N, N'')$, which sends the class of E' to the class of E''. If $N'' = N'$ and i is an inner automorphism of N, then the map induced by i is the identity. Thus we have a functor $\mathcal{E}(N, -)$ on the category of topological groups and isomorphisms between them up to conjugacy.

Let M be a topological group and N be a commutative topological group with an action of M. Write $H^2(M, N)$ for the subset of $\mathcal{E}(M, N)$ consisting of the classes of those extensions for which the action on N through M by conjugation coincides with the given action. The set $H^2(M, N)$ is functorial in N, with $f : N \to N'$ inducing the map that sends the class of E to the class of the quotient of $N' \rtimes E$ by N embedded as the set of $(f(n)^{-1}, n)$. Further $H^2(M, -)$ preserves finite products, so that $H^2(M, N)$ is an abelian group. When M is profinite and N is discrete and commutative, the abelian group $H^2(M, N)$ reduces to the usual one.

Given a short exact sequence

$$1 \to N' \to N \to N'' \to 1 \tag{19.7}$$

of topological groups with an action of M, we define as follows a long exact cohomology sequence

$$\begin{aligned} 1 \to H^0(M, N') \to H^0(M, N) \to H^0(M, N'') \to \\ \to H^1(M, N') \to H^1(M, N) \to H^1(M, N'') \to \mathcal{E}(M, N')/N'', \end{aligned} \tag{19.8}$$

where N'' acts on $\mathcal{E}(M, N')$ through its action on N' by automorphisms up to conjugacy, and the distinguished point of $\mathcal{E}(M, N')/N''$ is the class of $N' \rtimes M$. The image of n'' in N'' fixed by M under connecting map

$$H^0(M, N'') \to H^1(M, N')$$

is defined by choosing an n in N above n'' and taking the class of the section $m \mapsto c(m)m$, where the action of m on N sends n to $nc(m)$. The image of the class of the section s under the connecting map

$$H^1(M, N'') \to \mathcal{E}(M, N')/N''$$

is the class of the pullback along $s : M \to N'' \rtimes M$ of the extension

$$1 \to N' \to N \rtimes M \to N'' \rtimes M \to 1. \tag{19.9}$$

The other maps in (19.8) are defined by functoriality. The long exact sequence (19.8) is functorial for those morphisms from the short exact sequence (19.7) which are the isomorphisms on N'. When N' is central in N, the action of N' on $\mathcal{E}(M, N')$ is trivial, and the pointed set $\mathcal{E}(M, N')/N''$ in (19.8) may be replaced by its pointed subset $H^2(M, N')$. When M is profinite, N, N' and N'' are discrete, and N' is central, (19.8) with $\mathcal{E}(M, N')/N''$ replaced by $H^2(M, N')$ reduces to the usual long exact sequence.

Let G be a k-group of multiplicative type. By the universal property of $\pi_{\mathrm{mult}}(X)$, discarding the action of $\pi_{\mathrm{mult}}(X)$ defines an isomorphism from the category of principal $(\pi_{\mathrm{mult}}(X), G)$-bundles over X to the category of principal G-bundles over X, and hence a natural bijection

$$H^1_{\pi_{\mathrm{mult}}(X)}(X, G) \xrightarrow{\sim} H^1(X, G).$$

If k is algebraically closed, it follows from this and Lemma 7.5 that the functor $H^1(X,-)$ on k-groups of multiplicative type is representable. The representability in this case can also be deduced directly from the represesentable functor theorem, using the fact that by Lemma 11.1(i) and the exact cohomology sequence, $H^1(X,-)$ is left exact.

Let M be a Galois submodule of $\mathrm{Pic}(X_{\overline{k}})$. By the representability of $H^1(X_{\overline{k}},-)$ on $\overline{k}$-groups of multiplicative type, there is a unique

$$\tau \in H^1(X_{\overline{k}}, D(M)_{\overline{k}})$$

such that the push forward of τ along χ_μ is μ for every μ in M. We call τ the *tautological element*. For M' a Galois submodule of M, the projection from $D(M)_{\overline{k}}$ onto $D(M')_{\overline{k}}$ respects the tautological elements. The functor $H^1(X_{\overline{k}},-)$ on $\overline{k}$-groups of multiplicative type is represented by $D(\mathrm{Pic}(X_{\overline{k}}))_{\overline{k}}$ with universal element the tautological element.

For any Galois submodule M of $\mathrm{Pic}(X_{\overline{k}})$, the tautological element of $H^1(X_{\overline{k}}, D(M)_{\overline{k}})$ is fixed by the action of $\mathrm{Gal}(\overline{k}/k)$ through $\overline{k}$.

The action of $\mathrm{Gal}(\overline{k}/k)$ on the set $G(\overline{k})$ of $\overline{k}$-points of an affine k-group G through its action on $\overline{k}$ will be called the *standard action*. It is continuous for the Krull topology on $G(\overline{k})$. The cohomology sets

$$H^n(\mathrm{Gal}(\overline{k}/k), G(\overline{k}))$$

for $n = 0, 1$, and when G is commutative for $n = 2$, will be understood to be those for which the action of $\mathrm{Gal}(\overline{k}/k)$ on $G(\overline{k})$ is standard. Similarly when G is commutative, extensions of $\mathrm{Gal}(\overline{k}/k)$ by $G(\overline{k})$ will be understood to those for which the action of $\mathrm{Gal}(\overline{k}/k)$ is standard.

Let F be a commutative transitive affine groupoid over $\overline{k}$. Then the F-scheme F^{diag} is constant, with

$$F^{\mathrm{diag}} = Z(F)_{\overline{k}}.$$

The action of $\mathrm{Gal}(\overline{k}/k)$ defined by the extension $F(\overline{k})_{\overline{k}}$ of $\mathrm{Gal}(\overline{k}/k)$ by

$$F^{\mathrm{diag}}(\overline{k})_{\overline{k}} = Z(F)(\overline{k})$$

can be seen as follows to be the standard action. By (11.3), conjugation by v in $F(\overline{k})_{\overline{k}}$ above σ in $\mathrm{Gal}(\overline{k}/k)$ sends d in $F^{\mathrm{diag}}(\overline{k})_{\overline{k}}$ to

$$v \circ {}^{\sigma}d \circ v^{-1},$$

where $\circ$ denotes the composition and $^{-1}$ the inverse in the groupoid F. On the other hand σ acting on $Z(F)(\overline{k})$ through $\overline{k}$ sends d to the $\overline{k}$-point over $\overline{k}$ of the constant scheme F^{diag} corresponding to the $\overline{k}$-point ${}^{\sigma}d$ over σ. Again this is the conjugate in F of ${}^{\sigma}d$ by any v above σ.

Let G be a commutative affine k-group. If F is a commutative transitive affine groupoid over $\overline{k}$ and

$$j : G \xrightarrow{\sim} Z(F)$$

be an isomorphism of k-groups, then the push forward of the extension $F(\overline{k})_{\overline{k}}$ of $\mathrm{Gal}(\overline{k}/k)$ by $F^{\mathrm{diag}}(\overline{k})_{\overline{k}}$ along the topological isomorphism from $F^{\mathrm{diag}}(\overline{k})_{\overline{k}}$ to $G(\overline{k})$ induced by j^{-1} is by the above a topological extension of $\mathrm{Gal}(\overline{k}/k)$ by $G(\overline{k})$. By Proposition 11.4, we obtain in this way an equivalence from the category of pairs (F, j) to the category topological extensions of $\mathrm{Gal}(\overline{k}/k)$ by $G(\overline{k})$. Indeed the essential surjectivity follows because a $\overline{k}$-homomorphism $G_{\overline{k}} \to G'_{\overline{k}}$ descends to k

provided that the induced homomorphism $G(\overline{k}) \to G'(\overline{k})$ commutes with the action of $\mathrm{Gal}(\overline{k}/k)$. In particular, by assigning to (F,j) the class

$$[F,j] \in H^2(\mathrm{Gal}(\overline{k}/k), G(\overline{k}))$$

of its corresponding extension, we obtain a bijection from the set of isomorphism classes of pairs (F,j) to $H^2(\mathrm{Gal}(\overline{k}/k), G(\overline{k}))$.

Let G be a commutative affine k-group and P be a principal $G_{\overline{k}}$-bundle over $X_{\overline{k}}$. Given a pair (F,j) as above, a right action of F on P over X will be said to be compatible with j if its restriction to F^{diag} coincides modulo $j_{\overline{k}}$ with the given action of $G_{\overline{k}}$ on P. The right action action then defines a structure of principal F-bundle over X on P. A pair (F,j) with a right action of F on P over X compatible with j can exist only if

$$[P] \in H^1(X_{\overline{k}}, G_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}. \tag{19.10}$$

This follows from Lemma 11.3, because if P_0 and P_1 are the pullbacks of P along two $\overline{k}$-points s_0 and s_1 of $\mathrm{Spec}(\overline{k})$, then any $\overline{k}$-point of F above (s_0,s_1) defines an isomorphism of principal $G_{\overline{k}}$-bundles from P_0 to P_1.

Suppose now that (19.10) holds. It can be seen as follows that there exists a transitive affine groupoid I_P over $\overline{k}/k$ whose points in a k-scheme S above (s_0,s_1) are the isomorphisms from P_{s_1} to P_{s_0} of principal G_S-bundles over X_S. Writing G as the limit of its k-quotients of finite type, we reduce to the case where G is of finite type. There is then a finite Galois subextension k_1 of $\overline{k}$ such that P is the pullback of a principal G_{k_1}-bundle over X_{k_1}. Factoring through

$$\Gamma = \mathrm{Spec}(\overline{k}) \times_k \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k_1) \times_k \mathrm{Spec}(k_1)$$

and using the fact that $[P]$ is fixed by $\mathrm{Gal}(\overline{k}/k)$ then shows that the pullbacks of P onto Γ along the two projections are isomorphic as principal G_Γ-bundles over X_Γ. Choosing an isomorphism and using (19.1) and the finiteness conditions on X now shows that I_P exists and is isomorphic as a scheme over Γ to G_Γ.

The action of $G_{\overline{k}}$ on P induces a $\overline{k}$-isomorphism $G_{\overline{k}} \xrightarrow{\sim} {I_P}^{\mathrm{diag}}$, so that I_P is commutative. The $\overline{k}$-isomorphism descends to a k-isomorphism

$$i_P : G \xrightarrow{\sim} Z(I_P).$$

We have a canonical right action of ${I_P}^{\mathrm{op}}$ on P over X, which is compatible with i_P. By construction, an action of (F,j) on P over X which is compatible with j is the same as an isomorphism, from (F,j) to $({I_P}^{\mathrm{op}}, i_P)$. An isomorphism from P to P' of principal $G_{\overline{k}}$-bundles over $X_{\overline{k}}$ defines in an evident way an isomorphism from (I_P, i_P) to $(I_{P'}, i_{P'})$.

It follows from the above that if (19.10) holds, then pairs (F,j) exist for which there is a right action of F on P over X compatible with j, and any two of them are isomorphic: they are the ones isomorphic to $({I_P}^{\mathrm{op}}, i_P)$. We thus have a map

$$H^1(X_{\overline{k}}, G_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)} \to H^2(\mathrm{Gal}(\overline{k}/k), G(\overline{k})) \tag{19.11}$$

which sends the class of P to $[F,j]$, where (F,j) is such that there is a right action of F on P over X compatible with j. It is natural in G, and hence a homomorphism of abelian groups. When G is of finite type, (19.11) is a differential in the Hochschild–Serre spectral sequence.

The fibre of (19.11) above $[F,j]$ is the image of the map

$$H^1(X,F) \to H^1(X_{\overline{k}}, G_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)} \tag{19.12}$$

obtained by restricting to the action of the diagonal of F and then along $j_{\overline{k}}$. The group $H^1(k, G)$ of automorphisms of (F, j) up to conjugacy acts freely on $H^1(X, F)$, with orbits the fibres of (19.12). The action is free because by (19.1) and the finiteness condition on X, any automorphism of a principal $G_{\overline{k}}$-bundle over $X_{\overline{k}}$ is induced by a $\overline{k}$-point of G. Equivalently, by (11.17) with $H = X$, the fibre of (19.11) above the class of E is the image of the map

$$H^1(X, G_{\overline{k}}, E) \to H^1(X_{\overline{k}}, G_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)} \tag{19.13}$$

obtained by discarding the action of E. The group $H^1(\mathrm{Gal}(\overline{k}/k), G(\overline{k}))$ of automorphisms of the extension E up to conjugacy acts freely on $H^1(X, G_{\overline{k}}, E)$, with orbits the fibres of (19.13).

Suppose now that X is a Severi–Brauer variety over k of dimension n. Fix an isomorphism of $\overline{k}$-schemes

$$i : \mathbf{P}^n{}_{\overline{k}} \xrightarrow{\sim} X_{\overline{k}}.$$

With the standard action of PGL_{n+1} on $\mathbf{P}^n$, we have a morphism

$$[\overline{k}] \to (PGL_{n+1})_{[\overline{k}]} \tag{19.14}$$

of groupoids over $\overline{k}$ which sends the point (s_0, s_1) in a k-scheme S to the point of PGL_{n+1} in S that acts as the automorphism ${i_{s_0}}^{-1} \circ i_{s_1}$ of $\mathbf{P}^n{}_S$ over S. The corresponding homomorphism

$$\mathrm{Gal}(\overline{k}/k) \to PGL_{n+1}(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k)$$

sends σ to the $\overline{k}$-point of PGL_{n+1} that acts as the automorphism of $\mathbf{P}^n{}_{\overline{k}}$ over $\overline{k}$ given by composing the pullback of i along σ with i^{-1}. Its class in

$$H^1(\mathrm{Gal}(\overline{k}/k), PGL_{n+1}(\overline{k}))$$

is that of the principal PGL_{n+1}-bundle $\underline{\mathrm{Iso}}_k(\mathbf{P}^n, X)$ over k.

The pullback of (19.14) along the constant morphism

$$(GL_{n+1})_{[\overline{k}]} \to (PGL_{n+1})_{[\overline{k}]}$$

of groupoids over $\overline{k}$ is the embedding of a commutative transitive affine subgroupoid F of $(GL_{n+1})_{[\overline{k}]}$ over $\overline{k}$. The embedding into GL_{n+1} of its centre $\mathbf{G}_m$ then defines an isomorphism j from $\mathbf{G}_m$ to $Z(F)$. The class

$$[F, j] \in H^2(\mathrm{Gal}(\overline{k}/k), \mathbf{G}_m(\overline{k})) = H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}^*)$$

is the image under connecting map associated to the short exact sequence

$$1 \to \mathbf{G}_m \to GL_{n+1} \to PGL_{n+1} \to 1$$

of the class of $\underline{\mathrm{Iso}}_k(\mathbf{P}^n, X)$. It is thus the class in the Brauer group of k of the Severi–Brauer variety X over k.

Write Q for the affine space $\mathbf{A}^{n+1}$ with the origin removed. There is a canonical projection from Q to $\mathbf{P}^n$, and the standard linear action of GL_{n+1} on $\mathbf{A}^{n+1}$ induces an action on Q above the action of PGL_{n+1} on $\mathbf{P}^n$. The action of $\mathbf{G}_m$ on Q for which g acts as the homothety g^{-1} defines on Q a structure of principal $\mathbf{G}_m$-bundle over X of degree 1, whose associated line bundle is Q completed with the hyperplane at infinity as zero section.

Let P be a principal $\mathbf{G}_{m\overline{k}}$-bundle over $X_{\overline{k}}$ of degree 1. It can be seen as follows that with (F,j) as above there exists a right action of F on P over X compatible with j. We may suppose that $P = Q_{\overline{k}}$ with structural morphism

$$Q_{\overline{k}} \to \mathbf{P}^n_{\overline{k}} \xrightarrow{i} X_{\overline{k}}.$$

The action of GL_{n+1} on Q defines an action of $(GL_{n+1})_{[\overline{k}]}$ on $Q_{\overline{k}}$. Restricting to F, we obtain an action of F on $Q_{\overline{k}}$ above the trivial action on $X_{\overline{k}}$. It defines a right action of F^{op} on $Q_{\overline{k}}$ over X compatible with $-j$. Since the inverse involution of F defines an isomorphism from $(F^{\mathrm{op}},-j)$ to (F,j), this gives the required right action of F.

The degree defines an isomorphism from $\mathrm{Pic}(X_{\overline{k}})$ to the trivial Galois module $\mathbf{Z}$. Thus

$$D(\mathrm{Pic}(X_{\overline{k}})) = \mathbf{G}_m,$$

with the tautological class in $H^1(X_{\overline{k}},(\mathbf{G}_m)_{\overline{k}})$ the class of a principal $(\mathbf{G}_m)_{\overline{k}}$-bundle over $X_{\overline{k}}$ of degree 1. It follows that with $G = \mathbf{G}_m$ in (19.11), the image of the tautological class in the Brauer group of k is the class of the Severi–Brauer variety X. Let E be a topological extension of $\mathrm{Gal}(\overline{k}/k)$ by $\overline{k}^*$ with class in the Brauer Group of k that of X. Then since $H^1(k,\mathbf{G}_m)$ is trivial By Hilbert's Theorem 90, the map (19.13) with $G = \mathbf{G}_m$ is injective, so that there is a unique elemement in $H^1(X,(\mathbf{G}_m)_{\overline{k}},E)$ whose image in $H^1(X_{\overline{k}},(\mathbf{G}_m)_{\overline{k}})$ is of degree 1.

Theorem 19.3. *Let X be a smooth geometrically connected projective curve of genus 0 over k, and E be a topological extension of $\mathrm{Gal}(\overline{k}/k)$ by $\overline{k}^*$ with class in the Brauer group $H^2(\mathrm{Gal}(\overline{k}/k),\overline{k}^*)$ of k that of X. Then there exists a unique element α of $H^1(X,(\mathbf{G}_m)_{\overline{k}},E)$ with image under*

$$H^1(X,(\mathbf{G}_m)_{\overline{k}},E) \to H^1(X_{\overline{k}},(\mathbf{G}_m)_{\overline{k}}) = \mathbf{Z}$$

of degree 1. The functor $H^1(X,-,-)$ on reductive Galois extended $\overline{k}$-groups up to conjugacy is represented by $((\mathbf{G}_m)_{\overline{k}},E)$ with universal element α.

Proof. The existence and uniqueness of α have been seen above. To prove the representation statement, we reduce by Corollary 16.16 to the case where $k = \overline{k}$ is algebraically closed. In that case X has a k-point x and $H^1(X,-) = \widetilde{H}^1(X,x,-)$, so that the result follows from Theorem 19.2. $\square$

20. Curves of genus 1

In this section k is a field of characteristic 0 and $\overline{k}$ is an algebraic closure of k.

In this section we describe principal bundles with reductive structure group over a curve X over k of genus 1. The study of such bundles is reduced by Theorem 19.1 below to the study of the universal groupoid $\pi_{\text{étm}}(X)$ of proétale by multiplicative type of Section 10. The main results are Theorem 20.7, which deals with the case where X has a k-point, and Theorem 20.8, which deals with the general case.

Lemma 20.1. *Let X be a geometrically connected proétale cover of a smooth projective curve of genus 1 over k. Then $H^1(X,\mathcal{O}_X)$ is 1-dimensional over k.*

Proof. We may write X as the filtered limit of geometrically connected finite étale covers X_λ of a smooth projective curve X_0 of genus 1 over k. If X_λ is the spectrum of the $\mathcal{O}_{X_0}$-algebra $\mathcal{R}_\lambda$, then $H^1(X,\mathcal{O}_X)$ is the colimit of the filtered system of

1-dimensional k-vector spaces $H^1(X_0, \mathcal{R}_\lambda)$. Since by Lemma 9.1(ii) the transition homomorphisms are injective, the result follows. $\square$

Lemma 20.2. *Let X be a geometrically connected proétale cover of a smooth projective curve of genus 1 over k. Then non-split extensions of $\mathcal{O}_X$ by $\mathcal{O}_X$ exist, and any two of them are isomorphic as $\mathcal{O}_X$-modules. If $\mathcal{E}$ is such a non-split extension, then every symmetric power $S^r\mathcal{E}$ of $\mathcal{E}$ is indecomposable.*

Proof. The existence and uniqueness statements follow from Lemma 20.1.

To prove that $S^r\mathcal{E}$ is indecomposable for $r = 1$, suppose the contrary. Then $\mathcal{E}$ is the direct sum of line bundles $\mathcal{E}'$ and $\mathcal{E}''$. We may assume $H^0(X, \mathcal{E}'^\vee)$ is 0, because $\mathcal{E}^\vee$ is a non-split extension of $\mathcal{O}_X$ by $\mathcal{O}_X$ and hence $H^0(X, \mathcal{E}^\vee)$ is 1-dimensional. The epimorphism from $\mathcal{E}$ to $\mathcal{O}_X$ then induces an epimorphism and hence an isomorphism from $\mathcal{E}''$ to $\mathcal{O}_X$. Thus $\mathcal{E}$ is isomorphic to ${\mathcal{O}_X}^2$, which is impossible.

To prove that $S^r\mathcal{E}$ is indecomposable for arbitrary r, we may suppose that k is algebraically closed and that X is simply connected. It then suffices to use the example in Remark 15.7. $\square$

We denote by n_G the multiplication by the integer n on a commutative group scheme G over k. Let X be an abelian variety over k, with base point x. We obtain a k-pointed geometric universal cover $(\widetilde{X}, \widetilde{x})$ of (X, x) by taking the filtered limit of copies of X indexed by the integers, with transition morphisms the n_X. Since $\mathrm{Pic}(\widetilde{X}_{\overline{k}})$ is the colimit of copies of $\mathrm{Pic}(X_{\overline{k}})$, with n_X acting as n on $\mathrm{Pic}^0(X_{\overline{k}})$ and as n^2 on $NS(X_{\overline{k}})$, the group $\mathrm{Pic}(\widetilde{X}_{\overline{k}})$ is uniquely divisible, and the homomorphism from $\mathrm{Pic}(X_{\overline{k}})$ to $\mathrm{Pic}(\widetilde{X}_{\overline{k}})$ induces an isomorphism

$$\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{Pic}(\widetilde{X}_{\overline{k}}) \tag{20.1}$$

of Galois modules.

Lemma 20.3. *Suppose that k is algebraically closed. Let X be a simply connected proétale cover of an elliptic curve over k, and $\mathcal{E}$ be a non-split extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. Then any indecomposable vector bundle over X rank $r+1$ is isomorphic to $\mathcal{L} \otimes_{\mathcal{O}_X} S^r\mathcal{E}$ for a line bundle $\mathcal{L}$ over X, determined uniquely up to isomorphism.*

Proof. Let $\mathcal{V}$ be an indecomposable vector bundle over X of rank $r+1$. If a line bundle $\mathcal{L}$ over X with

$$\mathcal{V} \xrightarrow{\sim} \mathcal{L} \otimes_{\mathcal{O}_X} S^r\mathcal{E}$$

exists, it is is unique by (20.1), because $\mathcal{L}^{\otimes(r+1)}$ must be isomorphic to the determinant of $\mathcal{V}$. To prove that $\mathcal{L}$ exists, we may suppose by (20.1) that $\mathcal{V}$ has trivial determinant. There is an elliptic curve X_0 over k such that $\mathcal{V}$ and $\mathcal{E}$ are the pullback along a morphism $X \to X_0$ of a vector bundle $\mathcal{V}_0$ over X_0 and a non-split extension $\mathcal{E}_0$ of $\mathcal{O}_{X_0}$ by $\mathcal{O}_{X_0}$. Then $\mathcal{V}_0$ is indecomposable, with determinant of degree 0 by (20.1). By Lemma 20.2 and [Ati57, Theorem 5], $\mathcal{V}_0$ is isomorphic to $\mathcal{L}_0 \otimes_{\mathcal{O}_{X_0}} S^r\mathcal{E}_0$ for a line bundle $\mathcal{L}_0$ over X_0. $\square$

Theorem 20.4. *Let X be a geometrically connected proétale cover of a smooth projective curve of genus 1 over k, and P be the push forward along an embedding $\mathbf{G}_a \to SL_2$ of a non-trivial principal $\mathbf{G}_a$-bundle over X. Then $\pi_{\text{étm}}(X) \times_{[X]} \mathrm{Iso}_{SL_2}(P)$ is universally reductive over X.*

Proof. We reduce first using Corollary 15.16 to the case where k is algebraically closed and then using Proposition 18.5 to the case where X is simply connected. Then $\pi_{\text{étm}}(X)$ coincides with $\pi_{\text{mult}}(X)$, and it is to be shown that

$$K = \pi_{\text{mult}}(X) \times_{[X]} \mathrm{Iso}_{SL_2}(P)$$

is universally reductive over X. Restricting to a k-point of X and using Lemma 3.2 shows that the indecomposable representations of K are the tensor products of those of $\pi_{\text{mult}}(X)$ with those of $\mathrm{Iso}_{SL_2}(P)$. The indecomposable representations of $\pi_{\text{mult}}(X)$ are those of rank 1, and passage to the underlying $\mathcal{O}_X$-module defines an isomorphism from the group of isomorphism classes such representations to $\mathrm{Pic}(X)$. If V is the standard representation of SL_2, then the indecomposable representations of $\mathrm{Iso}_{SL_2}(P)$ are the symmetric powers of $P \times^G V$, which has underlying $\mathcal{O}_X$-module a non-trivial extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. By Lemmas 20.2 and 20.3, condition (b) of Corollary 15.5 is thus satisfied. □

Let X be an abelian variety over k with base point x. Then for any k-point y of X, there is a unique translation automorphism T_y of the k-scheme X which sends x to y. By functoriality of $\pi_{\text{étm}}(X,x)$ in the pointed k-scheme (X,x), we may thus identify the fibres $\pi_{\text{étm}}(X,y)$ of the diagonal of $\pi_{\text{étm}}(X)$ with $\pi_{\text{étm}}(X,x)$.

The k-pointed geometric universal cover $(\widetilde{X},\widetilde{x})$ of (X,x) is a limit of copies of (X,x). Thus we may regard $\widetilde{X}$ as a commutative k-group with identity $\widetilde{x}$. The projection then defines a short exact sequence of commutative k-groups

$$1 \to \lim_n \mathrm{Ker}\, n_X \to \widetilde{X} \to X \to 1$$

which is functorial in X. It follows that

$$\pi_{\text{ét}}(X,x) = \lim_n \mathrm{Ker}\, n_X \tag{20.2}$$

is commutative. Passing to topological Galois modules of $\overline{k}$-points, we have

$$\pi_{\text{ét}}(X,x)(\overline{k}) = \lim_n X(\overline{k})_n$$

where the subscript n denotes the subgroup of elements annulled by n.

Since $\pi_{\text{ét}}(X,x)$ is commutative, it is a k-quotient of $\pi_{\text{mult}}(X,x) = D(\mathrm{Pic}(X_{\overline{k}}))$. The projection factors through an isomorphism

$$D(\mathrm{Pic}(X_{\overline{k}})_{\text{tors}}) \xrightarrow{\sim} \pi_{\text{ét}}(X,x), \tag{20.3}$$

defined by the property that it sends the tautological element to the class of the pointed principal $\pi_{\text{ét}}(X,x)$-bundle $(\widetilde{X},\widetilde{x})$. On Galois modules of characters, (20.3) induces the isomorphism that sends the character χ of $\pi_{\text{ét}}(X,x)$ over $\overline{k}$ to the element of $\mathrm{Pic}(X_{\overline{k}})_{\text{tors}}$ given by pushing forward along χ the class of $\widetilde{X}_{\overline{k}}$.

We may identify $\pi_{\text{étm}}(\widetilde{X},\widetilde{x}) = \pi_{\text{mult}}(\widetilde{X},\widetilde{x})$ with $D(\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}})$ using (20.1). The embedding of (19.6) then becomes

$$D(\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}}) \to \pi_{\text{étm}}(X,x). \tag{20.4}$$

The projection from $\pi_{\text{étm}}(X,x)$ onto $\pi_{\text{mult}}(X,x)$ is

$$\pi_{\text{étm}}(X,x) \to D(\mathrm{Pic}(X_{\overline{k}})). \tag{20.5}$$

Both (20.4) and (20.5) are natural in the abelian variety X. The composite of (20.4) with (20.5) is the k-homomorphism induced by the canonical homomorphism from $\mathrm{Pic}(X_{\overline{k}})$ to $\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}}$. Conjugation by a point of $\pi_{\text{étm}}(X,x)$ thus induces the

identity on $D(\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}})$ because it induces the identity on $D(\mathrm{Pic}(X_{\overline{k}}))$. It follows that (20.4) is the embedding of a central k-subgroup.

Let M be a Galois submodule of $\mathrm{Pic}(X_{\overline{k}})$ containing $\mathrm{Pic}(X_{\overline{k}})_{\mathrm{tors}}$. Then for a quotient M' of $\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}}$ we have a short exact sequence of Galois modules

$$0 \to M \to \mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}} \to M' \to 0.$$

We obtain from (19.6) using (20.3) a central extension of k-groups

$$1 \to D(M') \to \pi_{\text{étm}}(X,x) \to D(M) \to 1, \tag{20.6}$$

where the first arrow is the restriction of (20.4) to $D(M')$, and the second is (20.5) composed with the projection onto $D(M)$. The second arrow sends the universal element to the tautological element.

Taking $M = \mathrm{Pic}^0(X_{\overline{k}})$ and $M' = NS(X_{\overline{k}})_{\mathbf{Q}}$ in (20.6), we obtain a central extension of k-groups

$$1 \to D(NS(X_{\overline{k}})_{\mathbf{Q}}) \to \pi_{\text{étm}}(X,x) \to D(\mathrm{Pic}^0(X_{\overline{k}})) \to 1 \tag{20.7}$$

which is functorial in X. The k-endomorphism n_{X*} of $\pi_{\text{étm}}(X,x)$ induces the nth power on $D(\mathrm{Pic}^0(X_{\overline{k}}))$ and the n^2th power on $D(NS(X_{\overline{k}})_{\mathbf{Q}})$.

The commutative k-groups $D(\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}})$ and $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ are uniquely divisible, in the sense that the nth power morphism is an isomorphism for every $n \neq 0$, because $\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}}$ and $NS(X_{\overline{k}})_{\mathbf{Q}}$ are $\mathbf{Q}$-vector spaces. Rational powers of any point of these k-groups are thus uniquely defined. Similarly for $n \neq 0$ the k-group $D(\mathrm{Pic}^0(X_{\overline{k}}))$ has no n-torsion, in the sense that the nth power morphism is a monomorphism, because $\mathrm{Pic}^0(X_{\overline{k}})$ is divisible.

Let G' and G'' be commutative affine k-groups. We write $\mathrm{Alt}^2(G', G'')$ for the abelian group of those morphisms of k-schemes

$$z : G' \times_k G' \to G''$$

which are bilinear, i.e. $z(g', -)$ and $z(-, g')$ are group homomorphisms for every point g' of G', and alternating, i.e. $z(g', g') = 1$ for every g'. To every central extension G of G' by G'' is associated its *commutator morphism* in $\mathrm{Alt}^2(G', G'')$, given by factoring through $G' \times_k G'$ the commutator of G, which sends the point (g,h) of $G \times_k G$ to the point

$$[g,h] = ghg^{-1}h^{-1}$$

of the k-subgroup G'' of G.

We denote by

$$c_X \in \mathrm{Alt}^2(D(\mathrm{Pic}^0(X_{\overline{k}})), D(NS(X_{\overline{k}})_{\mathbf{Q}}))$$

the commutator morphism associated to the central extension (20.7). By (20.6), the cokernel of (20.4) is $D(\mathrm{Pic}(X_{\overline{k}})_{\mathrm{tors}})$ and the kernel of (20.5) is $D(NS(X_{\overline{k}}) \otimes \mathbf{Q}/\mathbf{Z})$. Since (20.4) is the embedding of a central k-subgroup and (20.5) is the projection onto the maximal commutative k-quotient, c_X thus factors as

$$\begin{array}{ccc}
D(\mathrm{Pic}^0(X_{\overline{k}})) \times_k D(\mathrm{Pic}^0(X_{\overline{k}})) & \xrightarrow{c_X} & D(NS(X_{\overline{k}})_{\mathbf{Q}}) \\
{\scriptstyle r \times r}\downarrow & & \uparrow{\scriptstyle v} \\
D(\mathrm{Pic}(X_{\overline{k}})_{\mathrm{tors}}) \times_k D(\mathrm{Pic}(X_{\overline{k}})_{\mathrm{tors}}) & \longrightarrow & D(NS(X_{\overline{k}}) \otimes \mathbf{Q}/\mathbf{Z})
\end{array}$$

where r is the projection and v is the embedding.

The right arrow v of the above diagram is the embedding of the smallest k-subgroup through which c_X factors. It follows from this that if j_1 and j_2 are k-homomorphisms from $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ to a commutative affine k-group G, and if G has no n-torsion for some $n > 1$, then the implication

$$j_1 \circ c_X = j_2 \circ c_X \implies j_1 = j_2 \tag{20.8}$$

holds. Indeed suppose $j_1 \circ c_X = j_2 \circ c_X$. Then $j_1 \circ v = j_2 \circ v$ because c_X and hence v factors through the equaliser of j_1 and j_2. Since the cokernel of v is the k-torus $D(NS(X_{\overline{k}}))$, the image in G of the difference of j_1 and j_2 is a k-subtorus with no n-torsion and hence trivial.

The describe the commutator c_X explicitly, it is enough to determine its composite with the character $\chi_{\nu/n}$ of $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ over $\overline{k}$ for every ν in $NS(X_{\overline{k}})$ and $n \neq 0$. This will be done in Proposition 20.5 below, using the Weil pairing $\widetilde{e}_n$ and the morphism φ_ν defined by ν from $X_{\overline{k}}$ to its dual. Recall (e.g. [Mum70, p.183]) that if $\widehat{X}$ the dual abelian variety to X, the Weil pairing associated to X is the pairing

$$\widetilde{e}_n : X(\overline{k})_n \times \widehat{X}(\overline{k})_n \to (\overline{k}^*)_n$$

of Galois modules defined as follows. If we regard $X_{\overline{k}}$ is a principal $(\operatorname{Ker} n_{X_{\overline{k}}})$-bundle over itself, its class in H^1 the tautological element, and λ in $\widehat{X}(\overline{k})_n$ is the push forward of this class along the character χ_λ of $\operatorname{Ker} n_{X_{\overline{k}}}$. Then

$$\widetilde{e}_n(a, \lambda) = \chi_\lambda(a)^{-1}.$$

Given ν in $NS(X_{\overline{k}})$, we write

$$\varphi_\nu : X_{\overline{k}} \to \widehat{X}_{\overline{k}}$$

for the morphism over $\overline{k}$ that sends a to the class of $(T_a)^*\mathcal{L} \otimes \mathcal{L}^\vee$, where $\mathcal{L}$ is a line bundle over X with class ν.

The proof of Proposition 20.5 is based on the behavior of the groups $\pi_{\text{étm}}(X, x)$ under translation, which we now describe.

By (19.3), the functor $\widetilde{H}^1(X, x, -)$ on k-groups of proétale by multiplicative type up to conjugacy is represented by $\pi_{\text{étm}}(X, x)$ with universal element the class of $\pi_{\text{étm}}(X)_{-,x}$. Let y be a k-point of X. Then the translation automorphism T_y of the k-scheme X extends uniquely to an automorphism

$$(T_y, \theta_y) : (X, \pi_{\text{étm}}(X)) \xrightarrow{\sim} (X, \pi_{\text{étm}}(X))$$

of the groupoid in k-schemes $(X, \pi_{\text{étm}}(X))$. A k-point t of $\pi_{\text{étm}}(X)$ such that

$$(d_0(t), d_1(t)) = (y, x) \tag{20.9}$$

exists if and only if $\pi_{\text{étm}}(X)_{-,x}$ has a k-point above y if and only if $(T_y)^*\pi_{\text{étm}}(X)_{-,x}$ has a k-point above x. Given such a t, define a k-automorphism ξ_t of $\pi_{\text{étm}}(X, x)$ by

$$\xi_t(g) = t^{-1}\theta_y(g)t. \tag{20.10}$$

It is independent, up to conjugacy, of the choice of t. Since ξ_t is an isomorphism, the push forward $\xi_{t*}\pi_{\text{étm}}(X)_{-,x}$ of $\pi_{\text{étm}}(X)_{-,x}$ along ξ_t is simply $\pi_{\text{étm}}(X)_{-,x}$ with the action $p \mapsto p\xi_t^{-1}(g)$ of the point g of $\pi_{\text{étm}}(X, x)$. Thus we have an isomorphism $p \mapsto \theta_y(p)t$ from $\xi_{t*}\pi_{\text{étm}}(X)_{-,x}$ to $\pi_{\text{étm}}(X)_{-,x}$ over T_y which is compatible with the actions of $\pi_{\text{étm}}(X, x)$. It follows that we have an isomorphism

$$\xi_{t*}\pi_{\text{étm}}(X)_{-,x} \xrightarrow{\sim} (T_y)^*\pi_{\text{étm}}(X)_{-,x}$$

of principal $\pi_{\text{étm}}(X,x)$-bundles over X. For G of proétale by multiplicative type, we thus have a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_k(\pi_{\text{étm}}(X,x),G)/G(k) & \xrightarrow{\sim} & \widetilde{H}^1(X,x,G) \\ {\scriptstyle \mathrm{Hom}_k(\xi_t,G)/G(k)}\big\downarrow & & \big\downarrow{\scriptstyle (T_y)^*} \\ \mathrm{Hom}_k(\pi_{\text{étm}}(X,x),G)/G(k) & \xrightarrow{\sim} & \widetilde{H}^1(X,x,G) \end{array} \tag{20.11}$$

with the horizontal arrows the natural bijections defined by the class of $\pi_{\text{étm}}(X)_{-,x}$.

For any n, the endomorphism n_X of X extends uniquely to an endomorphism

$$(n_X,\theta_n) : (X,\pi_{\text{étm}}(X)) \xrightarrow{\sim} (X,\pi_{\text{étm}}(X))$$

of the groupoid in k-schemes $(X,\pi_{\text{étm}}(X))$. Suppose that $n \neq 0$. Then may regard X as an étale cover of itself with structural morphism n_X, and $\pi_{\text{étm}}(X)$ acts uniquely on X over itself. If s is a point of $\pi_{\text{étm}}(X)$ with $d_1(s) = x$, then applying (19.2) to this étale cover shows that there is a unique point t of $\pi_{\text{étm}}(X)$ with $d_1(t) = x$ such that $\theta_n(t) = s$. Further $d_0(t) = sx$.

For $n \neq 0$ we have a short exact sequence of k-groups

$$1 \to \pi_{\text{étm}}(X,x) \xrightarrow{n_{X*}} \pi_{\text{étm}}(X,x) \xrightarrow{r_n} \mathrm{Ker}\, n_X \to 1, \tag{20.12}$$

where r_n is the projection. Let y be an element of $X(k)_n = (\mathrm{Ker}\, n_X)(k)$, and s be a k-point of $\pi_{\text{étm}}(X,x)$ with

$$r_n(s) = y.$$

Then there a unique k-point t of $\pi_{\text{étm}}(X)$ with $d_1(t) = x$ such that

$$\theta_n(t) = s. \tag{20.13}$$

Now the fibre $\pi_{\text{étm}}(X,x)$ of $\pi_{\text{étm}}(X)$ above (x,x) acts on X over itself through the action of $\mathrm{Ker}\, n_X$ by translation. Thus $sx = y$, so that (20.9) holds. If we regard $\pi_{\text{étm}}(X,x)$ as a normal k-subgroup of itself embedded by n_{X*}, then conjugation by s^{-1} defines a k-automorphism ζ_s of $\pi_{\text{étm}}(X,x)$, so that

$$n_{X*}(\zeta_s(g)) = s^{-1} n_{X*}(g) s. \tag{20.14}$$

Since n_{X*} is the restriction of θ_n to $\pi_{\text{étm}}(X,x)$, it follows from (20.10) that

$$\zeta_s = \xi_t.$$

In particular the square (20.11) commutes with ζ_s for ξ_t.

Proposition 20.5. *Let X be an abelian variety over k and ν be an element of $NS(X_{\overline{k}})$. Then*

$$\chi_{\nu/n}(c_X(a,a')) = \widetilde{e}_n(a_n,\varphi_\nu(a'_n))^{-1}$$

for any $n \neq 0$ and $\overline{k}$-points a and a' of $D(\mathrm{Pic}^0(X_{\overline{k}}))$ with respective images a_n and a'_n in $D(\mathrm{Pic}(X_{\overline{k}})_n) = \mathrm{Ker}\, n_X$.

Proof. We may suppose that k is algebraically closed. If we take $G = \mathbf{G}_m$ in (20.11), and identify characters of $D(\mathrm{Pic}(X))$ with characters of $\pi_{\text{étm}}(X,x)$ by inflation, then the inverses of the horizontal isomorphisms of (20.11) send the element μ of

$$\widetilde{H}^1(X,x,\mathbf{G}_m) = \mathrm{Pic}(X)$$

to the character χ_μ of $\pi_{\text{étm}}(X,x)$. Suppose that μ lies above ν, and let g and g' be k-points of $\pi_{\text{étm}}(X,x)$ above a and a'. Then with r_n as in (20.12) we have

$r_n(g') = a'_n$. The square (20.11) thus commutes with ξ_t replaced by $\zeta_{g'}$ and y by a'_n. Hence

$$\chi_\mu(\zeta_{g'}(g)) = \chi_{(T_{a'_n})^*\mu}(g).$$

Multiplying with $\chi_\mu(g^{-1})$ gives

$$\chi_\mu(\zeta_{g'}(g)g^{-1}) = \chi_{\varphi_\nu(a'_n)}(g) = \widetilde{e}_n(a_n, \varphi_\nu(a'_n))^{-1}.$$

Since g'^{-1} commutes with $n_{X*}(g)g'n_{X*}(g)^{-1}$, we have by (20.14)

$$n_{X*}(\zeta_{g'}(g)g^{-1}) = g'^{-1}n_{X*}(g)g'n_{X*}(g)^{-1} = [n_{X*}(g), g'].$$

Since n_{X*} is a monomorphism which acts as the nth power on $D(\mathrm{Pic}^0(X))$ and the n^2th power on $D(NS(X)_{\mathbf{Q}})$, it follows that $\zeta_{g'}(g)g^{-1}$ lies in $D(NS(X)_{\mathbf{Q}})$ and

$$\zeta_{g'}(g)g^{-1} = c_X(a^n, a')^{1/n^2} = c_X(a, a')^{1/n}.$$

Thus $\chi_\mu(\zeta_{g'}(g)g^{-1}) = \chi_{\nu/n}(c_X(a, a'))$. □

It follows from Proposition 20.5 that the largest k-subgroup G of $D(\mathrm{Pic}^0(X_{\overline{k}}))$ for which $c_X(-, G)$ is trivial is its identity component

$$D(\mathrm{Pic}^0(X_{\overline{k}})/\,\mathrm{Pic}^0(X_{\overline{k}})_{\mathrm{tors}}) = D(\mathrm{Pic}^0(X_{\overline{k}})_{\mathbf{Q}}).$$

Indeed if ν the class of an ample line bundle on X, then given a' and n with $a'_n \neq 0$, we have $\varphi_\nu(a'_m) \neq 0$ for some multiple m of n, and hence $\widetilde{e}_m(a_m, \varphi_\nu(a'_m)) \neq 1$ for some a. The centre of $\pi_{\text{étm}}(X, x)$ thus coincides with its identity component, or equivalently with the image of the embedding (20.4).

Let G' and G'' be affine k-groups. Given z in $\mathrm{Alt}^2(G', G'')$, we write

$$C_z$$

for the affine k-group with underlying k-scheme $G'' \times_k G'$ and product given by

$$(g'', g')(h'', h') = (g''h''z(g', h'), g'h').$$

The embedding $g'' \mapsto (g'', 1)$ of G'' into C_z and the projection onto G' define on C_z a structure of extension of G' by G'' with commutator morphism z^2. Given also commutative affine k-groups G'_1 and G''_1 and z_1 in $\mathrm{Alt}^2(G'_1, G''_1)$, any pair $j' : G' \to G'_1$ and $j'' : G'' \to G''_1$ of k-homomorphisms such that

$$z_1 \circ (j' \times_k j') = j'' \circ z$$

induces a k-homomorphism $C_z \to C_{z_1}$ with underlying morphism of k-schemes $j'' \times_k j'$. In particular for any integer n we write

$$n_z : C_z \to C_z$$

for the k-endomorphism induced by the pair $n_{G'}$ and $(n^2)_{G''}$.

A commutative affine k-group G will be called uniquely 2-divisible if 2_G is an isomorphism. Every point g of G has then a unique square root $g^{1/2}$.

Let G be an affine k-group equipped with an involution, i.e. a k-automorphism ι with $\iota^2 = 1_G$. Suppose that the k-subgroup G^ι of invariants is central and uniquely 2-divisible and that G/G^ι is commutative. Then $[\iota(g), g]$ lies in G^ι for every point g of G, so that

$$[\iota(g), g] = \iota([\iota(g), g]) = [g, \iota(g)] = [\iota(g), g]^{-1}$$

and hence $[\iota(g), g] = 1$. Thus $\iota(g)$ commutes with g, so that $g\iota(g)$ lies in G^ι. It follows that g can be written uniquely in the form

$$g = g_+g_-$$

with $\iota(g_+) = g$ and $\iota(g_-) = g^{-1}$. Indeed $g_+ = (g\iota(g))^{1/2}$. If c in $\mathrm{Alt}^2(G/G^\iota, G^\iota)$ is the commutator and u in $\mathrm{Alt}^2(G/G^\iota, G^\iota)$ is given by

$$u(g', h') = c(g', h')^{1/2},$$

we then have an isomorphism

$$(G, \iota) \xrightarrow{\sim} (C_u, (-1)_u) \tag{20.15}$$

of k-groups with an involution, which sends g to $(g_+, \overline{g})$ with $\overline{g}$ the image of g in G/G^ι. The isomorphism (20.15) is natural in (G, ι). It is the unique k-homomorphism from G to C_u which is compatible with both the involutions and the structures of extension of G/G^ι by G^ι.

Now let X be an abelian variety over k with base point x. Define

$$u_X \in \mathrm{Alt}^2(D(\mathrm{Pic}^0(X_{\overline{k}})), D(NS(X_{\overline{k}})_{\mathbf{Q}}))$$

by

$$u_X(g', h') = c_X(g', h')^{1/2}$$

The involution $(-1)_{X*}$ of $\pi_{\text{étm}}(X, x)$ induces through the short exact sequence (20.7) the identity on $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ and the inverse involution on $D(\mathrm{Pic}^0(X_{\overline{k}}))$. Since $D(\mathrm{Pic}^0(X_{\overline{k}}))$ has no 2-torsion, it follows that

$$\pi_{\text{étm}}(X, x)^{(-1)_{X*}} = D(NS(X_{\overline{k}})_{\mathbf{Q}}).$$

Thus (20.15) with $G = \pi_{\text{étm}}(X, x)$ and $\iota = (-1)_{X*}$ gives by (20.7) an isomorphism

$$\pi_{\text{étm}}(X, x) \xrightarrow{\sim} C_{u_X} \tag{20.16}$$

of k-groups with involution. It is natural in X, by naturality of (20.15) and functoriality of (20.7), and is the unique k-homomorphism compatible with the involutions and the structures of extension of $D(\mathrm{Pic}^0(X_{\overline{k}}))$ by $D(NS(X_{\overline{k}})_{\mathbf{Q}})$.

Proposition 20.6. *Let X be an abelian variety with base point x. Then there is a unique element of $H^1(X, x, C_{u_X})$ which has image in $H^1(X, x, D(\mathrm{Pic}^0(X_{\overline{k}})))$ the tautological element and is fixed by the involution $((-1)_X, (-1)_{u_X})$ of (X, C_{u_X}). For any element α of $H^1(X, x, C_{u_X})$ with image in $H^1(X, x, D(\mathrm{Pic}^0(X_{\overline{k}})))$ the tautological element, the functor $H^1(X, x, -)$ on k-groups of proétale by multiplicative type is represented by C_{u_X} with universal element α.*

Proof. Let j be a k-homomorphism from $\pi_{\text{étm}}(X, x)$ to C_{u_X}. Then the image α_j of j under the bijection

$$\mathrm{Hom}_k(\pi_{\text{étm}}(X, x), C_{u_X}) \xrightarrow{\sim} H^1(X, x, C_{u_X})$$

defined by the universal property of $\pi_{\text{étm}}(X, x)$ lies above the tautological element if and only if j is compatible with the projections onto $D(\mathrm{Pic}^0(X_{\overline{k}}))$. When this condition holds, j is also compatible with the embeddings of $D(NS(X_{\overline{k}})_{\mathbf{Q}})$, because j then induces an endomorphism j_1 of $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ which satisfies $j_1 \circ c_X = c_X$ and hence is the identity by (20.8). Since α_j is universal if and only if j is an isomorphism, the second statement is clear. By (19.5) with $f = (-1)_X$ and $G = C_{u_X}$, the element α_j is fixed by the involution $((-1)_X, (-1)_{u_X})$ if and only if j is compatible with the involutions $(-1)_{X*}$ and $(-1)_{u_X}$. Thus (20.16) is the unique j for which α_j lies above the tautological element and is fixed by $((-1)_X, (-1)_{u_X})$. $\square$

We may identify $\pi_{\text{étm}}(X,x)$ with C_{u_X} using (20.16). With this identification, the universal element in

$$H^1(X,x,C_{u_X})$$

is the unique element of Proposition 20.6 which is fixed by $((-1)_X,(-1)_{u_X})$ and which has image in $H^1(X,x,D(\mathrm{Pic}^0(X_{\overline{k}})))$ the tautological element, because by (19.5) the universal element in $H^1(X,x,\pi_{\text{étm}}(X,x))$ is fixed by $((-1)_X,(-1)_{X*})$. Similarly, by naturality of (20.16), the k-homomorphism $f_*:C_{u_X}\to C_{u_{X'}}$ defined by a commutative diagram of the form (19.5) for a morphism $f:(X,x)\to(X',x')$ of abelian varieties coincides with that defined by functoriality of C_{u_X}. In particular

$$n_{X*}=n_{u_X}$$

for any n.

With the identification of $\pi_{\text{étm}}(X,x)$ and C_{u_X}, we have a commutative diagram

$$\begin{array}{ccccc} D(\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}}) & \xrightarrow{e} & C_{u_X} & \xrightarrow{p} & D(\mathrm{Pic}(X_{\overline{k}})) \\ \uparrow & & \| & & \downarrow \\ D(NS(X_{\overline{k}})_{\mathbf{Q}}) & \longrightarrow & C_{u_X} & \longrightarrow & D(\mathrm{Pic}^0(X_{\overline{k}})) \end{array} \tag{20.17}$$

where e is (20.4), p is (20.5), the bottom row is (20.7), the left vertical arrow is the embedding and the right vertical arrow is the projection. It is functorial in X. By compatibility of (20.16) with the extensions, the bottom row of (20.17) is the canonical structure of extension defined by C_{u_X}. The k-homomorphism $p\circ e$ is that defined by the canonical homomorphism from $\mathrm{Pic}(X_{\overline{k}})$ to $\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}}$.

When applied with $G=D(\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}})$ and ι defined by $(-1)_X$, (20.15) is the decomposition

$$D(\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}})=D(NS(X_{\overline{k}})_{\mathbf{Q}})\times_k D(\mathrm{Pic}^0(X_{\overline{k}})_{\mathbf{Q}}),$$

where $NS(X_{\overline{k}})_{\mathbf{Q}}$ is identified with the Galois submodule of $\mathrm{Pic}(X_{\overline{k}})_{\mathbf{Q}}$ of invariants under $(-1)_X$, and the projections are defined by the embeddings. Since e in (20.17) is compatible with the involutions, it is uniquely determined by the induced k-homomorphisms from $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ to $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ and from $D(\mathrm{Pic}^0(X_{\overline{k}})_{\mathbf{Q}})$ to $D(\mathrm{Pic}^0(X_{\overline{k}}))$. The first is the identity, by the left square of (20.17), and the second is that defined by the canonical homomorphism from $\mathrm{Pic}^0(X_{\overline{k}})$ to $\mathrm{Pic}^0(X_{\overline{k}})_{\mathbf{Q}}$, by the right square.

Since the k-subgroup of invariants of $D(\mathrm{Pic}(X_{\overline{k}}))$ under the involution induced by $(-1)_X$ is the torus $D(NS(X_{\overline{k}}))$, and hence not uniquely 2-divisible, it is less trivial to determine p in (20.17) explicitly. To do so, note first that by (20.17), the restriction of p to the k-subgroup $D(NS(X_{\overline{k}})_{\mathbf{Q}})$ of C_{u_X} is defined by the canonical homomorphism from $\mathrm{Pic}(X_{\overline{k}})$ to $NS(X_{\overline{k}})_{\mathbf{Q}}$. It thus suffices to determine the restriction

$$p_-:D(\mathrm{Pic}^0(X_{\overline{k}}))\to D(\mathrm{Pic}(X_{\overline{k}}))$$

of p to the k-subscheme $D(\mathrm{Pic}^0(X_{\overline{k}}))$ of C_{u_X}. This k-subscheme consists of those points s of C_{u_X} that $(-1)_{u_X}$ sends to s^{-1}. If X is of dimension >0, it is not a k-subgroup of C_{u_X}, and in fact p_- will not be a k-homomorphism.

To determine p_-, we use the decomposition

$$D(\mathrm{Pic}(X_{\overline{k}}))=D(\mathrm{Pic}(X_{\overline{k}})^{(-1)_X})\times_{D(\mathrm{Pic}(X_{\overline{k}})_2)} D(\mathrm{Pic}^0(X_{\overline{k}})),$$

with the projections defined by the embeddings of the Galois submodules into $\mathrm{Pic}(X_{\overline{k}})$. This decomposition holds because $\mathrm{Pic}^0(X_{\overline{k}})$ is 2-divisible, so that above every element of $NS(X_{\overline{k}})$ there exists an element of $\mathrm{Pic}(X_{\overline{k}})$ fixed by $(-1)_X$. By the right square of (20.17), the compomenent of p_- at $D(\mathrm{Pic}^0(X_{\overline{k}}))$ is the identity. It remains to determine the component

$$p_{+-} : D(\mathrm{Pic}^0(X_{\overline{k}})) \to D(\mathrm{Pic}(X_{\overline{k}})^{(-1)_X})$$

of p_- at $D(\mathrm{Pic}(X_{\overline{k}})^{(-1)_X})$.

To determine p_{+-}, define for each λ in $\mathrm{Pic}(X_{\overline{k}})^{(-1)_X}$ a map of Galois sets

$$\varepsilon_\lambda : X(\overline{k})_2 \to \{\pm 1\} \subset \overline{k}^*,$$

as follows. Let $\mathcal{L}$ be a line bundle over $X_{\overline{k}}$ with class λ. Then there is a unique involution i of $\mathcal{L}$ above $(-1)_{X_{\overline{k}}}$ which acts as $+1$ on $\mathcal{L}_x$. Now $X(\overline{k})_2$ is the fixed point set of $X(\overline{k})$ under $(-1)_X$. We define $\varepsilon_\lambda(y)$ as the action of i on $\mathcal{L}_y$. In general, ε_λ is not a group homomorphism. Since $D(\mathrm{Pic}(X)_2)$ is a finite étale k-scheme with Galois set of k-points $X(\overline{k})_2$, and since $\varepsilon_\lambda(y)$ is additive in λ for given y, there is a unique morphism of k-schemes

$$\varepsilon : D(\mathrm{Pic}(X_{\overline{k}})_2) \to D(\mathrm{Pic}(X_{\overline{k}})^{(-1)_X})$$

such that

$$\chi_\lambda(\varepsilon(y)) = \varepsilon_\lambda(y)$$

for each y in $X(\overline{k})_2$. We now show that p_{+-} factors as

$$D(\mathrm{Pic}^0(X_{\overline{k}})) \to D(\mathrm{Pic}(X_{\overline{k}})_2) \xrightarrow{\varepsilon} D(\mathrm{Pic}(X_{\overline{k}})^{(-1)_X}),$$

where the first arrow is the projection.

To see that p_{+-} factors as above, we may suppose that k is algebraically closed. Let s be a k-point of $D(\mathrm{Pic}^0(X_{\overline{k}}))$ with image y in $D(\mathrm{Pic}(X_{\overline{k}})_2)$. It is to be shown that for every λ in $\mathrm{Pic}(X)^{(-1)_X}$ we have

$$\chi_\lambda(p_{+-}(s)) = \varepsilon_\lambda(y).$$

Let $\mathcal{L}$ and i be as above, and write $\mathcal{L}^*$ for the the complement of the zero section of $\mathcal{L}$, regarded as a principal $\mathbf{G}_m$-bundle over X. After equipping $\mathcal{L}^*$ with a base k-point above x, the class of $\mathcal{L}^*$ is the push forward of the tautological element in $H^1(X, x, D(\mathrm{Pic}(X)^{(-1)_X}))$ along χ_λ, and hence the push forward of the universal element of $H^1(X, x, C_{u_X})$ along the composite

$$p_\lambda : C_{u_X} \to \mathbf{G}_m$$

of the component of p at $D(\mathrm{Pic}(X)^{(-1)_X})$ with χ_λ. There is thus a unique morphism $q_{\mathcal{L}} : \pi_{\text{étm}}(X)_{-,x} \to \mathcal{L}^*$ of schemes over X which preserves base k-points such that

$$(p_\lambda, q_{\mathcal{L}}) : (C_{u_X}, \pi_{\text{étm}}(X)_{-,x}) \to (\mathbf{G}_m, \mathcal{L}^*)$$

is compatible with the actions of $C_{u_X} = \pi_{\text{étm}}(X, x)$ and $\mathbf{G}_m$. Further $(1_X, p_\lambda, q_{\mathcal{L}})$ is compatible, by the universal property of C_{u_X}, with the involution of

$$(X, C_{u_X}, \pi_{\text{étm}}(X)_{-,x})$$

induced by $(-1)_X$ and the involution $((-1)_X, 1, i)$ of $(X, \mathbf{G}_m, \mathcal{L}^*)$. The involution of $(C_{u_X}, \pi_{\text{étm}}(X)_{-,x})$ defined by $(-1)_X$ is given by restriction of the involution θ_{-1} of $\pi_{\text{étm}}(X)$. By (20.13) with $n = 2$, we have

$$\theta_2(t) = s$$

for a k-point t of $\pi_{\text{étm}}(X)$ such that (20.9) holds. Thus t is a k-point of $\pi_{\text{étm}}(X)_{-,x}$ above y. Then $\theta_{-1}(t)$ is also a k-point of $\pi_{\text{étm}}(X)_{-,x}$ above y. We have $\theta_n(s) = s^n$, because θ_n restricted to C_{u_X} is n_{u_X}. Hence

$$\theta_2(\theta_{-1}(t)s) = \theta_2(\theta_{-1}(t))\theta_2(s) = \theta_{-1}(\theta_2(t))s^2 = \theta_{-1}(s)s^2 = s.$$

Since θ_2 is a momomorphism, it follows that

$$\theta_{-1}(t)s = t.$$

Applying $(p_\lambda, q_{\mathcal{L}})$, we have by compatibility with the actions and the involutions

$$i(q_{\mathcal{L}}(t))p_\lambda(s) = q_{\mathcal{L}}(t).$$

Since $\chi_\lambda(p_{+-}(s)) = p_\lambda(s)$, the required equality follows.

When X is an elliptic curve, $NS(X_{\overline{k}})$ is $\mathbf{Z}$ with the trivial Galois action. Thus $D(NS(X_{\overline{k}}) \otimes \mathbf{Q})$ is $D(\mathbf{Q})$, so that u_X is a k-morphism

$$u_X : D(\mathrm{Pic}^0(X_{\overline{k}})) \times_k D(\mathrm{Pic}^0(X_{\overline{k}})) \to D(\mathbf{Q}),$$

and $D(NS(X_{\overline{k}}) \otimes \mathbf{Q}/\mathbf{Z})$ is $D(\mathbf{Q}/\mathbf{Z}) = \lim_n \mu_n$.

Theorem 20.7. *Let X be an elliptic curve over k with base point x. Then there exists an element of $\widetilde{H}^1(X, x, C_{u_X})$ with image in*

$$\widetilde{H}^1(X, x, D(\mathrm{Pic}^0(X_{\overline{k}}))) = H^1(X, x, D(\mathrm{Pic}^0(X_{\overline{k}})))$$

the tautological element. If α is such an element, and if β is the image in

$$H^1(X, SL_2) = \widetilde{H}^1(X, x, SL_2)$$

of a non-zero element of $H^1(X, \mathbf{G}_a)$ under an embedding $\mathbf{G}_a \to SL_2$, then the functor $\widetilde{H}^1(X, x, -)$ on reductive k-groups up to conjugacy is represented by

$$C_{u_X} \times_k SL_2$$

with universal element (α, β).

Proof. The existence statement is clear, because the image α_0 in $\widetilde{H}^1(X, x, C_{u_X})$ of the universal element in $H^1(X, x, C_{u_X})$ lies above the tautological element in $\widetilde{H}^1(X, x, D(\mathrm{Pic}^0(X_{\overline{k}})))$. If also α lies above the tautological element, then by (19.3) and Proposition 20.6 there is a k-automorphism of C_{u_X} which sends α_0 to α. To prove the representation statement, we may thus suppose that $\alpha = \alpha_0$. Then α is the class of $\pi_{\text{étm}}(X)_{-,x}$. Let P be a principal SL_2-bundle over X with class β. Then (5.6) and Theorem 20.4 show that

$$\underline{\mathrm{Iso}}_{C_{u_X} \times SL_2}(\pi_{\text{étm}}(X)_{-,x} \times_X P) = \underline{\mathrm{Iso}}_{C_{u_X}}(\pi_{\text{étm}}(X)_{-,x}) \times_{[X]} \underline{\mathrm{Iso}}_{SL_2}(P)$$

is universally reductive over X. The representation statement now follows from Lemma 16.3. □

Let X be a k-scheme such that $X_{\overline{k}}$ is the underlying $\overline{k}$-scheme of an abelian variety over $\overline{k}$. Then X has a canonical structure of principal homogeneous space under its Albanese variety X_0. Thus X defines a class in the Weil–Châtelet group

$$\operatorname*{colim}_n H^1(\mathrm{Gal}(\overline{k}/k), X_0(\overline{k})_n) = H^1(\mathrm{Gal}(\overline{k}/k), X_0(\overline{k}))$$

of X_0. Since translation acts trivially on $\mathrm{Pic}^0((X_0)_{\overline{k}})$, we have a canonical isomorphism

$$\mathrm{Pic}^0(X_{\overline{k}}) \xrightarrow{\sim} \mathrm{Pic}^0((X_0)_{\overline{k}})$$

of Galois modules, given by the isomorphism $(X_0)_{\overline{k}} \xrightarrow{\sim} X_{\overline{k}}$ defined by any $\overline{k}$-point of X. Similarly we have a canonical isomorphism

$$NS(X_{\overline{k}}) \xrightarrow{\sim} NS((X_0)_{\overline{k}})$$

of Galois modules. We then have a morphism

$$u_X : D(\mathrm{Pic}^0(X_{\overline{k}})) \times D(\mathrm{Pic}^0(X_{\overline{k}})) \to D(NS(X_{\overline{k}}) \otimes \mathbf{Q})$$

which coincides, modulo these isomorphisms, with u_{X_0} as defined above.

For every $n \neq 0$ we have a short exact sequence

$$1 \to C_{u_X} \xrightarrow{n_{u_X}} C_{u_X} \to D(\mathrm{Pic}(X_{\overline{k}})_n) \to 1$$

of k-groups. Taking points in $\overline{k}$ gives a short exact sequence

$$1 \to C_{u_X}(\overline{k}) \xrightarrow{n_{u_X}} C_{u_X}(\overline{k}) \to X_0(\overline{k})_n \to 1 \tag{20.18}$$

of topological groups with an action of $\mathrm{Gal}(\overline{k}/k)$. Further for every $m \neq 0$ we have a morphism

$$\begin{tikzcd}
1 \arrow[r] & C_{u_X}(\overline{k}) \arrow[r, "(nm)_{u_X}"] & C_{u_X}(\overline{k}) \arrow[r] & X_0(\overline{k})_{nm} \arrow[r] & 1 \\
1 \arrow[r] & C_{u_X}(\overline{k}) \arrow[r, "n_{u_X}"] \arrow[u, equal] & C_{u_X}(\overline{k}) \arrow[r] \arrow[u, "m_z"'] & X_0(\overline{k})_n \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

of short exact sequences of topological groups with an action of $\mathrm{Gal}(\overline{k}/k)$. Using the functoriality of the connecting map in (19.8), we thus obtain a map

$$\operatorname*{colim}_n H^1(\mathrm{Gal}(\overline{k}/k), X_0(\overline{k})_n) \to \mathcal{E}(\mathrm{Gal}(\overline{k}/k), C_{u_X}(\overline{k}))/X_0(\overline{k})_{tors} \tag{20.19}$$

which assigns to each element of the Weil–Châtelet group of X_0 an orbit in the set of isomorphism classes of extensions of $\mathrm{Gal}(\overline{k}/k)$ by $C_{u_X}(\overline{k})$ under the action of $X_0(\overline{k})_{tors}$ on $C_{u_X}(\overline{k})$. In Theorem 20.8 below we consider instead of the target of (20.19) the coarser quotient

$$\mathcal{E}(\mathrm{Gal}(\overline{k}/k), C_{u_X}(\overline{k}))/\operatorname{Hom}_{\overline{k}}(D(\mathrm{Pic}^0(X_{\overline{k}}))_{\overline{k}}, D(NS(X_{\overline{k}}) \otimes \mathbf{Q})_{\overline{k}}) \tag{20.20}$$

by the group of all $\overline{k}$-automorphisms of $(C_{u_X})_{\overline{k}}$ over $D(\mathrm{Pic}^0(X_{\overline{k}}))_{\overline{k}}$. We obtain a canonical class in (20.20) by taking the image of the class of X in the Weil–Châtelet group of X_0 under (20.19) and then projecting onto (20.20).

For some $n \neq 0$, the class of X in the Weil–Châtelet group of X_0 is annulled by n. Then there is an isomorphism, compatible with the actions of X_0, from the push forward of X along n_{X_0} to X_0. Choosing such an isomorphism, we obtain a morphism

$$X \to X_0$$

compatible with the actions of X_0 and n_{X_0}. Restricting the action of X_0 on X to $\operatorname{Ker} n_{X_0}$ then gives a structure of principal $(\operatorname{Ker} n_{X_0})$-bundle over X_0 on X. The fibre X_{x_0} of X above the base point x_0 of X_0 is thus a principal homogeneous space under $\operatorname{Ker} n_{X_0}$. Let $\overline{x}$ be a $\overline{k}$-point of X above x_0. Then the class of X_{x_0} in $H^1(\mathrm{Gal}(\overline{k}/k), X_0(\overline{k})_n)$ is represented by the section

$$\mathrm{Gal}(\overline{k}/k) \to X_0(\overline{k})_n \rtimes \mathrm{Gal}(\overline{k}/k) \tag{20.21}$$

of topological groups that sends σ to (t, σ) with t defined by $\overline{x}t = {}^{\sigma}\overline{x}$.

The composition of $\pi_{\text{étm}}(X)$ defines a right action on the scheme $\pi_{\text{étm}}(X)_{-,\overline{x}}$ over $X_{\overline{k}}$ by the transitive affine groupoid

$$F = \pi_{\text{étm}}(X) \times_{[X]} [\mathrm{Spec}(\overline{k})]$$

over $\overline{k}$ given by pullback along the inclusion of $\overline{x}$. This action makes $\pi_{\text{étm}}(X)_{-,\overline{x}}$ a principal F-bundle over X. We have

$$F^{\mathrm{diag}} = \pi_{\text{étm}}(X_{\overline{k}}, \overline{x}) = (C_{u_X})_{\overline{k}},$$

and the class of the underlying principal F^{diag}-bundle over $X_{\overline{k}}$ of $\pi_{\text{étm}}(X)_{-,\overline{x}}$ has image in

$$H^1(X_{\overline{k}}, D(\mathrm{Pic}^0(X_{\overline{k}}))_{\overline{k}})$$

the tautological element.

The unique morphism

$$\pi_{\text{étm}}(X) \to \pi_{\text{étm}}(X_0) \times_{[X_0]} [X]$$

of groupoids over X induces by pullback along $[\mathrm{Spec}(\overline{k})] \to [X]$ a morphism

$$F \to \pi_{\text{étm}}(X_0, x_0)_{[\mathrm{Spec}(\overline{k})]} = (C_{u_X})_{[\mathrm{Spec}(\overline{k})]}$$

of groupoids over $\overline{k}$ whose restriction to the diagonal is

$$(n_{u_X})_{\overline{k}} : (C_{u_X})_{\overline{k}} \to (C_{u_X})_{\overline{k}}.$$

Passing to $\overline{k}$-points over $\overline{k}$ then gives by (19.2) a morphism

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C_{u_X}(\overline{k}) & \longrightarrow & F(\overline{k})_{\overline{k}} & \longrightarrow & \mathrm{Gal}(\overline{k}/k) & \longrightarrow & 1 \\
 & & \Big\| & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & C_{u_X}(\overline{k}) & \longrightarrow & C_{u_X}(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k) & \longrightarrow & X_0(\overline{k})_n \rtimes \mathrm{Gal}(\overline{k}/k) & \longrightarrow & 1
\end{array}$$

of short exact sequences of topological groups, where the left bottom arrow is defined by n_{u_X} and right vertical arrow is (20.21). By definition of the connecting map of (19.8), the image of the class of X_{x_0} in $H^1(\mathrm{Gal}(\overline{k}/k), X_0(\overline{k})_n)$ under the connecting map associated to (20.18) is thus the class in

$$\mathcal{E}(\mathrm{Gal}(\overline{k}/k), C_{u_X}(\overline{k}))/X_0(\overline{k})_n$$

of the extension $F(\overline{k})_{\overline{k}}$. It follows that the class of the extension $F(\overline{k})_{\overline{k}}$ in (20.20) is the canonical class.

Let E be a topological extension of $\mathrm{Gal}(\overline{k}/k)$ by $C_{u_X}(\overline{k})$ with class in (20.20) the canonical class. It can be seen as follows that $((C_{u_X})_{\overline{k}}, E)$ is a Galois extended $\overline{k}$-group, and that there exists an element in $H^1(X, (C_{u_X})_{\overline{k}}, E)$ with image under

$$H^1(X, (C_{u_X})_{\overline{k}}, E) \to H^1(X_{\overline{k}}, (C_{u_X})_{\overline{k}}) \to H^1(X_{\overline{k}}, D_{\overline{k}}(\mathrm{Pic}^0(X_{\overline{k}})))$$

the tautological element. It is enough to check this for *one* extension E. Choose an $X \to X_0$ and an x as above, and take $E = F(\overline{k})_{\overline{k}}$. Then $((C_{u_X})_{\overline{k}}, E)$ is the Galois extended $\overline{k}$-group associated to the groupoid F over $\overline{k}$, and the principal F-bundle $\pi_{\text{étm}}(X)_{-,\overline{x}}$ over X defines an element of $H^1(X, (C_{u_X})_{\overline{k}}, E)$ with the required property.

Theorem 20.8. *Let X be a geometrically connected smooth projective curve of genus 1 over k, and E be a topological extension of $\mathrm{Gal}(\overline{k}/k)$ by $C_{u_X}(\overline{k})$ with class in (20.20) the canonical class. Then $((C_{u_X})_{\overline{k}}, E)$ is a Galois extended $\overline{k}$-group, and there exists an element in $H^1(X, (C_{u_X})_{\overline{k}}, E)$ with image under*

$$H^1(X, (C_{u_X})_{\overline{k}}, E) \to H^1(X_{\overline{k}}, (C_{u_X})_{\overline{k}}) \to H^1(X_{\overline{k}}, D(\mathrm{Pic}^0(X_{\overline{k}}))_{\overline{k}})$$

the tautological element. If α is such an element, and if β is the image in

$$H^1(X, SL_2) = H^1(X, (SL_2)_{\overline{k}}, SL_2(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k))$$

of a non-zero element of $H^1(X, \mathbf{G}_a)$ under an embedding $\mathbf{G}_a \to SL_2$, then the functor $H^1(X, -, -)$ on reductive Galois extended $\overline{k}$-groups up to conjugacy is represented by

$$((C_{u_X})_{\overline{k}}, E) \times ((SL_2)_{\overline{k}}, SL_2(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k))$$

with universal element (α, β).

Proof. The existence statement has been proved above. To prove the representation statement, we reduce by Corollary 16.16 to the case where $k = \overline{k}$ is algebraically closed. In that case X has a k-point x and $H^1(X, -) = \widetilde{H}^1(X, x, -)$, so that the result follows from Theorem 20.7. □

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